On a space of compactoid filters

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Abstract

In his famous paper [3], Mizokami studied a nice mapping from the space of all continuous functions between topological spaces, endowed with compact-open topology, into the space of continuous functions between their Vietoris-hyperspaces, endowed with pointwise topology and proved, that it is indeed an embedding, whenever the base spaces are Hausdorff. In [2], this mapping is studied a little in the slightly more general context of almost arbitrary set-open topologies for sets of continuous functions between the base spaces. Here we give some results of a similar kind concerning the same natural mapping, for sets of continuous functions between the base spaces being equipped with the structure of continuous convergence. In order to do this, the families of compact subsets with Vietoris-hyperstructure are replaced by the families of the so called compactoid filters with a suitable pseudotopology.

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1 Basic notions and results

For a set X, we denote by $\mathfrak{P}(X)$ the power set of X and by $\mathfrak{P}_0(X)$ the power set without the empty set $\emptyset$. The set of all filters (resp. ultrafilters) on a set X is denoted by $\mathfrak{F}(X)$ (resp. $\mathfrak{F}_0(X)$), the set of all filters (resp. ultrafilters) refining a given filter φ on the same set X is denoted by $\mathfrak{F}(\varphi)$ (resp. $\mathfrak{F}_0(\varphi)$). A filter is not allowed to contain the empty set. By $\dot{x}$ we mean the singleton-filter, generated from the base consisting only of the singleton $\{x\}$.
With a topological space (X, τ) always a convergence structure q_τ is associated by $q_\tau := \{(\varphi, x) \in \mathfrak{F}(X) \times X|\ \varphi \supseteq \dot{x} \cap \tau\}$. By *neighbourhood filter* $\underline{U}(x)$ *of* x in a topological space (X, τ) we just mean the filter generated from $\dot{x} \cap \tau$.

1 Proposition
Let X, Y, I be sets, $\varphi, \chi_i \in \mathfrak{F}(X), i \in I$ and $f \in Y^X$. Then hold

(1) $A \in f(\varphi) \Longleftrightarrow f^{-1}(A) \in \varphi$ *and*

(2) $f(\bigcap_{i\in I} \chi_i) = \bigcap_{i\in I} f(\chi_i)$.

Proof: (1): Let $A \in f(\varphi)$, then $\exists B \in \varphi : f(B) \subseteq A$. Now, $f(B) \subseteq A \Leftrightarrow B \subseteq f^{-1}(A)$, implying $f^{-1}(A) \in \varphi$, if $A \in f(\varphi)$. The other direction is clear.
(2): $A \in f(\bigcap_{i\in I} \chi_i) \Leftrightarrow f^{-1}(A) \in \bigcap_{i\in I} \chi_i \Leftrightarrow \forall i \in I : f^{-1}(A) \in \chi_i \Leftrightarrow \forall i \in I : A \in$

$f(\chi_i) \Leftrightarrow A \in \bigcap_{i\in I} f(\chi_i)$. ■

For sets X, Y we will sometimes use the so called *evaluation map* ω, defined as

$$\omega : X \times Y^X \to Y : \omega(x, f) := f(x)$$

If $\mathcal{F}$ is a filter on Y^X and φ a filter on X, then by $\mathcal{F}(\varphi)$ we just mean $\omega(\varphi \times \mathcal{F})$, where $\varphi \times \mathcal{F}$ is the *product filter*, generated from all cartesian products of members of φ with members of $\mathcal{F}$.

2 Lemma
Let X, Y be sets, $\varphi \in \mathfrak{F}(X), \mathcal{F} \in \mathfrak{F}(Y^X)$. Then holds

$$\forall \psi \in \mathfrak{F}_0(\mathcal{F}(\varphi)) : \exists \mathcal{F}_0 \in \mathfrak{F}_0(\mathcal{F}), \varphi_0 \in \mathfrak{F}_0(\varphi) : \mathcal{F}_0(\varphi_0) \subseteq \psi\ .$$

Proof: Because of $\psi \supseteq \mathcal{F}(\varphi)$, each $C \in \psi$ has nonempty intersection with every $\omega(P \times F), P \in \varphi, F \in \mathcal{F}$, so for each $C \in \psi, P \in \varphi, F \in \mathcal{F}$, $\omega^{-1}(C) = \{(x, f) \in X \times Y^X |\ f(x) \in C\}$ has nonempty intersection with $P \times F$. Furthermore, for $C_1, C_2 \in \psi, P_1, P_2 \in \varphi, F_1, F_2 \in \mathcal{F}$ we have $\omega^{-1}(C_1) \cap \omega^{-1}(C_2) \supseteq \omega^{-1}(C_1 \cap C_2)$, which is not empty, because $C_1 \cap C_2 \in \psi$, and $(P_1 \times F_1) \cap (P_2 \times F_2) = (P_1 \cap P_2) \times (F_1 \cap F_2)$, with $P_1 \cap P_2 \in \varphi$ and $F_1 \cap F_2 \in \mathcal{F}$. Thus $\omega^{-1}(C_1) \cap (P_1 \times F_1) \cap \omega^{-1}(C_2) \cap (P_2 \times F_2) \neq \emptyset$, too. Now, $\mathfrak{B} := \{\omega^{-1}(C) \cap (P \times F)|\ C \in \psi, P \in \varphi, F \in \mathcal{F}\}$ is a filterbase on $X \times Y^X$ such that $[\mathrm{pr}_X(\mathfrak{B})] \supseteq \varphi$ and $[\mathrm{pr}_{Y^X}(\mathfrak{B})] \supseteq \mathcal{F}$, with the projection maps $\mathrm{pr}_X : X \times Y^X \to X : \mathrm{pr}_X((x, f)) := x$ and $\mathrm{pr}_{Y^X} : X \times Y^X \to Y^X : \mathrm{pr}_{Y^X}((x, f)) := f$, and $[\mathfrak{B}] \supseteq [\omega^{-1}(\psi)]$. Then there exists an ultrafilter $\mathfrak{B}_0$ on $X \times Y^X$, which contains $\mathfrak{B}$. This implies $[\omega(\mathfrak{B}_0)] \supseteq \omega(\mathfrak{B}) \supseteq \psi$, just meaning $[\omega(\mathfrak{B}_0)] = \psi$, because ψ is an ultrafilter. Now, define $\varphi_0 := \mathrm{pr}_X(\mathfrak{B}_0), \mathcal{F}_0 := \mathrm{pr}_{Y^X}(\mathfrak{B}_0)$. These are ultrafilters and we have $\varphi_0 \times \mathcal{F}_0 \subseteq \mathfrak{B}_0$, because $\forall P \times F \in \varphi_0 \times \mathcal{F}_0 : \exists B_P, B_F \in \mathfrak{B}_0 : P = \mathrm{pr}_X(B_P), F = \mathrm{pr}_{Y^X}(B_F) \Rightarrow \mathfrak{B}_0 \ni B_P \cap B_F \subseteq \mathrm{pr}_X(B_P \cap B_F) \times \mathrm{pr}_{Y^X}(B_P \cap B_F) \subseteq P \times F$. So, $\mathcal{F}_0(\varphi_0) = [\omega(\varphi_0 \times \mathcal{F}_0)] \subseteq \omega(\mathfrak{B}_0) = \psi$ follows. ■

3 Corollary
Let X, Y be sets, $\varphi \in \mathfrak{F}(X)$, $f \in Y^X$ and $\psi \in \mathfrak{F}_0(Y)$ with $\psi \supseteq f(\varphi)$. Then there exists an ultrafilter $\varphi_0 \in \mathfrak{F}_0(\varphi)$ with $f(\varphi_0) = \psi$.

4 Corollary
Let X, Y be sets, $\varphi \in \mathfrak{F}(X)$ and $f \in Y^X$.
Then $\mathfrak{F}_0(f(\varphi)) = f(\mathfrak{F}_0(\varphi))$ $(:= \{f(\psi)|\ \psi \in \mathfrak{F}_0(\varphi)\})$ holds.

5 Lemma
Let X be a set, $\mathfrak{A} \subseteq \mathfrak{P}(X)$ and $\varphi \in \mathfrak{F}(X)$. Assume, $\mathfrak{A}$ is closed under finite unions of its elements. Then holds

$$\varphi \cap \mathfrak{A} \neq \emptyset \iff \forall \psi \in \mathfrak{F}_0(\varphi) : \psi \cap \mathfrak{A} \neq \emptyset\ ,$$

i.e. a filter contains an $\mathfrak{A}$–set, iff each refining ultrafilter contains an $\mathfrak{A}$–set.

Proof: Suppose $\forall \psi \in \mathfrak{F}_0(\varphi) : \exists A_\psi \in \mathfrak{A} : A_\psi \in \psi$. Now, assume $\varphi \cap \mathfrak{A} = \emptyset$. From this automatically follows $X \notin \mathfrak{A}$.
Consider $\mathfrak{B} := \{X \setminus A |\ A \in \mathfrak{A}\}$. Because of the closedness of $\mathfrak{A}$ under finite unions, $\mathfrak{B}$ is closed under finite intersection of its elements, and $\emptyset \notin \mathfrak{B}$, because $X \notin \mathfrak{A}$. For any $F \in \varphi, B \in \mathfrak{B}$ we have $F \cap B \neq \emptyset$, because $F \cap B = \emptyset$ would imply $F \subseteq X \setminus B \in \mathfrak{A}$ and therefore $\varphi \cap \mathfrak{A} \neq \emptyset$. So, $\varphi \cup \mathfrak{B}$ is a subbase of a filter and consequently, there exists an ultrafilter ψ, containing $\varphi \cup \mathfrak{B}$, therefore containing φ and the complement of every $\mathfrak{A}$–set - in contradiction to $\forall \psi \in \mathfrak{F}_0(\varphi) : \psi \cap \mathfrak{A} \neq \emptyset$.
The other direction of the statement of the lemma is obvious. ■

The set

$$adh(\varphi) := q_\tau(\mathfrak{F}_0(\varphi)) = \{x \in X |\ \exists \varphi_0 \in \mathfrak{F}_0(\varphi) : (\varphi_0, x) \in q_\tau\}$$

is called the *adherence* of the filter φ.

6 Proposition
Let (X, τ) be a topological space and φ a filter on X. Then holds

$$adh(\varphi) = \bigcap_{A \in \varphi} \overline{A} \ ,$$

where $\overline{A}$ means the closure of A, and especially, $adh(\varphi)$ itself is closed w.r.t. τ.

Proof: Let $x \in adh(\varphi)$. Then $\exists \varphi_0 \in \mathfrak{F}_0(\varphi) : (\varphi_0, x) \in q_\tau$ holds, implying $\forall A \in \varphi : x \in \overline{A}$, thus $x \in \bigcap_{A \in \varphi} \overline{A}$.
Otherwise let $x \in \bigcap_{A \in \varphi} \overline{A}$. Then we have $\forall A \in \varphi, U \in \dot{x} \cap \tau : \overline{A} \cap U \neq \emptyset$ and consequently $A \cap U \neq \emptyset$, because of the closedness properties. Thus, the family $\mathfrak{B} := \{U \cap A |\ U \in \dot{x} \cap \tau, A \in \varphi\}$ is a base for a filter, which refines φ and converges to x, implying $x \in adh(\varphi)$. ■

2 Compactoid filters

7 Definition
Let (X, q) be a convergence space and $\varphi \in \mathfrak{F}(X)$. Then φ is said to be **compactoid** *w.r.t. q, iff*

$$\forall \varphi_0 \in \mathfrak{F}_0(\varphi), P \in \varphi : P \cap q(\varphi_0) \neq \emptyset \ ,$$

i.e. for every refining ultrafilter of φ, every member of φ contains an element, to which the ultrafilter converges.
The set of all compactoid filters on X w.r.t. q is denoted by $\mathfrak{C}(X)_q$, or, if no misunderstanding should be to aware, simply by $\mathfrak{C}(X)$.

Obviously, all compactly generated filters are compactoid, and - at least for pre-topological spaces - all neighbourhood-filters are compactoid, too.

8 Lemma
Let (X,τ) be a topological space and φ a filter on X. Then φ is compactoid w.r.t. q_τ, iff for every family $(O_i)_{i\in I}$ of τ-open subsets O_i of X

$$\bigcup_{i\in I} O_i \in \varphi \iff \exists n \in I\!N, i_1,...,i_n \in I : \bigcup_{k=1}^{n} O_{i_k} \in \varphi$$

holds.

Proof: Let $\varphi \in \mathfrak{F}(X)$ be compactoid and an arbitrary family $(O_i)_{i\in I}$ of open sets with $\bigcup_{i\in I} O_i \in \varphi$ be given. Assume $\forall J \subseteq I, card(J) \in I\!N : \bigcup_{j\in J} O_j \notin \varphi$, just meaning $\forall P \in \varphi : P \cap (\bigcup_{j\in J} O_j)^c \neq \emptyset$, consequently the filter-base $\mathfrak{B} := \{X \setminus \bigcup_{j\in J} O_j \mid J \subseteq I, card(J) \in I\!N\}$ is compatible with φ and so, there exists an ultrafilter φ_0, which contains both, φ and $\mathfrak{B}$. Then φ_0 converges especially on $\bigcup_{i\in I} O_i$ to a point x_0, because of the compactoidness of φ. Now, x_0 belongs to at least one of the open sets, say $x \in O_{i_x}$, which therefore must be contained in φ_0 - in contradiction to the fact, that φ_0 should contain $X \setminus O_{i_x}$ by construction.
Otherwise, let $\varphi \in \mathfrak{F}(X)$ be given with the property, that it contains a finite union of elements of every collection of open sets, whose union is contained in φ. Assume, there would exist a refining ultrafilter φ_0 on φ, which doesn't converge on some element P of φ. Then every point $p \in P$ has an open neighbourhood O_p, which is not contained in φ_0. But $\bigcup_{p\in P} O_p$ is an element of φ, because P is, and so there must exist a finite subset $\{p_1,...,p_n\}$ of P s.t. $\bigcup_{k=1}^{n} O_{p_k} \in \varphi \subseteq \varphi_0$. But then, because φ_0 is an ultrafilter, it must contain one of these O_{p_k} - a contradiction.■

9 Proposition
If $(X,\tau),(Y,\sigma)$ are topological spaces, $f \in C(X,Y)$ and $\varphi \in \mathfrak{C}(X)$, then $f(\varphi) \in \mathfrak{C}(Y)$.

Proof: If $\psi_0 \in \mathfrak{F}_0(f(\varphi))$, then there exists by corollary 3 an ultrafilter $\varphi_0 \in \mathfrak{F}_0(\varphi)$ with $f(\varphi_0) = \psi_0$. Because of the compactoidness of φ, this ultrafilter converges on every member of φ, thus by continuity of f, the image $f(\varphi_0) = \psi_0$ converges on every image $f(A)$ with $A \in \varphi$. But these images form a base for $f(\varphi)$. ■

3 Function Spaces

Let X and Y be sets and $A \subseteq X$, $B \subseteq Y$; then let be $(A,B) := \{f \in Y^X \mid f(A) \subseteq B\}$. Now let X be a set, (Y,σ) a topological space and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. Then the topology $\tau_{\mathfrak{A}}$ on Y^X (resp. $C(X,Y)$), which is defined by the open subbase $\{(A,W) \mid A \in \mathfrak{A}, W \in \sigma\}$ is called the **set–open topology, generated by $\mathfrak{A}$**, or shortly the **$\mathfrak{A}$–open topology** (see [9] (2.26)).
The structure of continuous convergence for sets of continuous functions is denoted

by q_c.

By $\mathfrak{F}(X)_{\mathfrak{A}}$ we denote the set of all filters on X, which have a base, consisting of elements of $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$.

10 Proposition
Let X be a set, (Y, σ) a topological space and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$, $\mathcal{F} \in \mathfrak{F}(Y^X)$, $f \in Y^X$. Then holds
$$(\mathcal{F}, f) \in q_{\tau_{\mathfrak{A}}} \iff \forall \varphi \in \mathfrak{F}(X)_{\mathfrak{A}} : \mathcal{F}(\varphi) \supseteq f(\varphi) \cap \sigma \ .$$

Proof: Let $(\mathcal{F}, f) \in q_{\tau_{\mathfrak{A}}}$ and $\varphi \in \mathfrak{F}(X)_{\mathfrak{A}}$. For any $W \in \sigma \cap f(\varphi)$ there is an $A \in \mathfrak{A}$, such that $f(A) \subseteq W$, because of $\varphi \in \mathfrak{F}(X)_{\mathfrak{A}}$. This means $f \in (A, W) \in \tau_{\mathfrak{A}}$, implying $(A, W) \in \mathcal{F}$ by $\mathcal{F} \xrightarrow{\tau_{\mathfrak{A}}} f$. So, we have $W \supseteq \omega(A, (A, W)) \in \mathcal{F}(\varphi)$.
If $\forall \varphi \in \mathfrak{F}(X)_{\mathfrak{A}} : \mathcal{F}(\varphi) \supseteq f(\varphi) \cap \sigma$ holds, we may chose the principal filters $[A]$ with $A \in \mathfrak{A}$ for φ to get $\mathcal{F}(A) \subseteq W$ for all $W \in \sigma \cap f(A)$, implying $(A, W) \in \mathcal{F}$ for any $A \in \mathfrak{A}, W \in \sigma$. ■

Now, we extend the class of the set–open topologies on $C(X, Y)$ to a greater class of convergence structures.

11 Definition
Let (X, τ), (Y, σ) be topological spaces and $\tilde{\mathfrak{A}} \subseteq \mathfrak{F}(X)$. Then we call
$$q_{\tilde{\mathfrak{A}}} := \left\{ (\mathcal{F}, f) \in \mathfrak{F}(C(X, Y)) \times C(X, Y) | \ \forall \varphi \in \tilde{\mathfrak{A}} : (\mathcal{F}(\varphi), f(\varphi)) \in \tilde{q_\sigma} \right\}$$
*the structure of $\tilde{\mathfrak{A}}$–**continuous convergence** for $C(X, Y)$.*

Obviously, every convergence $q_{\tau_{\mathfrak{A}}}$, generated from a set-open topology $\tau_{\mathfrak{A}}$ coincides with the structure of $\mathfrak{F}(X)_{\mathfrak{A}}$-continuous convergence on $C(X, Y)$, just by proposition 10.

To use the word "continuous" in definition 11, may be justified by the following.

12 Lemma
Let $(X, \tau), (Y, \sigma)$ be topological spaces and $\tilde{\mathfrak{A}} \subseteq \mathfrak{F}(X)$. Then holds:

(1) If all members of $\tilde{\mathfrak{A}}$ are compactoid, then $q_{\tilde{\mathfrak{A}}}$ is splitting, i.e. $q_c \subseteq q_{\tilde{\mathfrak{A}}}$.

(2) If $\tilde{\mathfrak{A}} \supseteq \{\underline{U}(x) | \ x \in X\}$, then $q_{\tilde{\mathfrak{A}}}$ is conjoining, i.e. $q_{\tilde{\mathfrak{A}}} \subseteq q_c$.

(3) If $\{\underline{U}(x) | \ x \in X\} \subseteq \tilde{\mathfrak{A}} \subseteq \mathfrak{C}(X)$, then $q_{\tilde{\mathfrak{A}}} = q_c$.

Proof: (1): Let $(\mathcal{F}, f) \in q_c$, $\varphi \in \tilde{\mathfrak{A}}$ and $V \in f(\varphi) \cap \sigma$. By lemma 2, for every $\psi_0 \in \mathfrak{F}_0(\mathcal{F}(\varphi))$, there are $\mathcal{F}_0 \in \mathfrak{F}_0(\mathcal{F}), \varphi_0 \in \mathfrak{F}_0(\varphi)$ such that $\mathcal{F}_0(\varphi_0) \subseteq \psi_0$. Now, $f^{-1}(V) \in \varphi$ and φ is compactoid, thus $\exists x_0 \in f^{-1}(V) : (\varphi_0, x_0) \in q_\tau$. This implies $(\mathcal{F}_0(\varphi_0), f(x_0)) \in q_\sigma$, because $\mathcal{F}$ converges continuously to f, and so $\mathcal{F}_0$ does. The

given V is an open neighbourhood of $f(x_0)$, thus $V \in \mathcal{F}_0(\varphi_0) \subseteq \psi_0$. So, every refining ultrafilter of $\mathcal{F}(\varphi)$ contains V and therefore $V \in \mathcal{F}(\varphi)$. This holds for all $V \in f(\varphi) \cap \sigma$, implying $\mathcal{F}(\varphi) \supseteq f(\varphi) \cap \sigma$. This is valid for all $\varphi \in \tilde{\mathfrak{A}}$, yielding $(\mathcal{F}, f) \in q_{\tilde{\mathfrak{A}}}$.
(2): Given $(\mathcal{F}, f) \in q_{\tilde{\mathfrak{A}}}$ and any $(\varphi, x) \in q_\tau$, we have $\varphi \supseteq \underline{U}(x)$, implying $\mathcal{F}(\varphi) \supseteq \mathcal{F}(\underline{U}(x)) \supseteq f(\underline{U}(x)) \cap \sigma$ by $\tilde{\mathfrak{A}}$-continuous convergence of $\mathcal{F}$ to f. By the continuity of f we get $f(\underline{U}(x)) \supseteq \overset{\bullet}{f(x)} \cap \sigma$, thus $\mathcal{F}(\varphi) \supseteq \overset{\bullet}{f(x)} \cap \sigma \cap \sigma = \overset{\bullet}{f(x)} \cap \sigma$, yielding $(\mathcal{F}(\varphi), f(x)) \in q_\sigma$ and now, because this holds for every $(\varphi, x) \in q_\tau$, we have $(\mathcal{F}, f) \in q_c$.
(3): follows immediately from (1) and (2), because the neighbourhood-filters are all compactoid. ■

4 Pseudotopologies for Sets of Filters

For $B \in \mathfrak{P}(X)$ and $\mathfrak{A} \subseteq \mathfrak{P}(X)$ we define $B^{-\mathfrak{a}} := \{A \in \mathfrak{A} | A \cap B \neq \emptyset\}$ (hit–set) and $B^{+\mathfrak{a}} := \{A \in \mathfrak{A} | A \cap B = \emptyset\}$ (miss–set). The upper Vietoris topology τ_V^+ on a family $\mathfrak{X} \subseteq \mathfrak{P}(X)$ of subsets of a topological space (X, τ) is generated by the subbase consisting of all $A^{+\mathfrak{x}}$ with closed $A \subseteq X$. The lower semifinite topology τ_l on $\mathfrak{X}$ is generated by the subbase of all $O^{-\mathfrak{x}}$ with open $O \subseteq X$. The Vietoris topology on a family $\mathfrak{X}$ is generated just from the upper Vietoris and the lower semifinite together.

For $\Phi \in \mathfrak{F}(\mathfrak{F}(X))$ we define

$$\Phi^{\uparrow} := \left\langle \left\{ \mathfrak{F}_0(\bigcap_{\chi \in \mathfrak{A}} \chi) \middle| \mathfrak{A} \in \Phi \right\} \right\rangle$$

which is a filter on $\mathfrak{F}_0(X)$. In case, that Φ is a filter on $\mathfrak{P}_0(X)$, we represent by the same symbol $\Phi^{\uparrow}$ just the filter on $\mathfrak{F}_0(X)$, which we get by mapping all nonempty subsets of X to their generated principal filters and then applying the $^{\uparrow}$-operator. Furthermore, for $\Phi \in \mathfrak{F}(\mathfrak{F}(X))$ we set

$$\Phi^{\cup\cap} := \bigcup_{\mathfrak{A} \in \Phi} \bigcap_{\varphi \in \mathfrak{A}} \varphi \ .$$

13 Proposition
Let X, Y be sets.

(1) *If Φ is an ultrafilter on $\mathfrak{F}_0(X)$, then $\Phi^{\cup\cap}$ is an ultrafilter on X.*

(2) *If Φ is a filter on $\mathfrak{F}_0(X)$ and $f \in Y^X$, then $f(\Phi^{\cup\cap}) \subseteq f(\Phi)^{\cup\cap}$ holds.*

Proof: (1): Let $A \in \mathfrak{P}(X)$. Then every ultrafilter on X either contains A or A^c. Thus $\mathfrak{F}_0(A) \cup \mathfrak{F}_0(A^c) = \mathfrak{F}_0(X)$, implying that either $\mathfrak{F}_0(A)$ or $\mathfrak{F}_0(A^c)$ is contained in Φ. But in the first case A and in the second case A^c belongs to $\Phi^{\cup\cap}$.
(2): From $A \in f(\Phi^{\cup\cap})$ we get $\exists \mathfrak{M} \in \Phi : A \in f(\bigcap_{\chi \in \mathfrak{M}} \chi)$ and we always have $f(\bigcap_{\chi \in \mathfrak{M}} \chi) \subseteq \bigcap_{\chi \in \mathfrak{M}} f(\chi)$, so we get $\exists \mathfrak{N}(:= f(\mathfrak{M})) \in f(\Phi) : A \in \bigcap_{\xi \in \mathfrak{N}} \xi$, just implying $A \in f(\Phi)^{\cup\cap}$. ■

14 Proposition
Let (X, τ) be a topological space, $\mathfrak{X} \subseteq \mathfrak{P}_0(X)$, $\Phi \in \mathfrak{F}(\mathfrak{X})$ and $A \in \mathfrak{X}$. Then for the upper Vietoris-topology τ_V^+ holds

$$(\Phi, A) \in q_{\tau_V^+} \Longleftrightarrow \forall \Phi_1 \in \mathfrak{F}_0(\Phi^{\uparrow}) : \exists \varphi \in \mathfrak{F}_0(A) : \Phi_1^{\cup\cap} \supseteq \varphi \cap \tau \ .$$

Proof: Assume, there would exist a $\Phi_1 \in \mathfrak{F}_0(\Phi^{\uparrow})$ s.t. $\forall \varphi \in \mathfrak{F}_0(A) : \exists U_\varphi \in \varphi \cap \tau : U_\varphi \notin \Phi_1^{\cup\cap}$, i.e. every ultrafilter φ on A contains a member of the family $\alpha := \tau \setminus \Phi_1^{\cup\cap}$. But this family is closed under finite unions because $\Phi_1^{\cup\cap}$ is an ultrafilter by proposition 13, yielding that a finite union of sets not contained in $\Phi_1^{\cup\cap}$ cannot be contained in $\Phi_1^{\cup\cap}$. So, lemma 5 applies and we get $[A] \cap \alpha \neq \emptyset$, i.e. $\exists O \in \tau : A \subseteq O \wedge O \notin \Phi_1^{\cup\cap}$. This implies $O^c \in \Phi_1^{\cup\cap}$, because $\Phi_1^{\cup\cap}$ is an ultrafilter, leading to $\mathfrak{F}_0(O^c) \in \Phi_1$ from which $(O^c)^{+\mathfrak{x}} \notin \Phi$ follows, thus $(\Phi, A) \notin q_{\tau_V^+}$.
Otherwise, let $(\Phi, A) \notin q_{\tau_V^+}$ be given. Then there exists an $O \in \tau$ with $A \in O$ and $(O^c)^{+\mathfrak{x}} \notin \Phi$. This means $\mathfrak{A} \setminus (O^c)^{+\mathfrak{x}} \neq \emptyset$ for all $\mathfrak{A} \in \Phi$, implying that $\{\{\varphi_0 \in \mathfrak{F}_0(X) |\ \exists K \in \mathfrak{A} : \varphi_0 \in \mathfrak{F}_0(K \setminus O)\} |\ \mathfrak{A} \in \Phi\}$ is a base for a filter on $\mathfrak{F}(X)$, which refines $\Phi^{\uparrow}$, and there must be an ultrafilter Φ_1, containing this base and therefore containing $\Phi^{\uparrow}$. But obviously, $O \notin \Phi_1^{\cup\cap}$, and so $\Phi_1^{\cup\cap}$ doesn't contain $\varphi \cap \tau$ for any ultrafilter φ on A. ■

15 Proposition
Let (X, τ) be a topological space, $\mathfrak{X} \subseteq \mathfrak{P}_0(X)$, $\Phi_0 \in \mathfrak{F}_0(\mathfrak{X})$ and $A \in \mathfrak{X}$. Then for the lower semifinite topology τ_l holds

$$(\Phi_0, A) \in q_{\tau_l} \Longleftrightarrow \ \forall \varphi \in \mathfrak{F}_0(A) : \exists \Phi_1 \in \mathfrak{F}_0(\Phi_0^{\uparrow}) : \Phi_1^{\cup\cap} \supseteq \varphi \cap \tau \ .$$

Proof: Let $\Phi_0 \in \mathfrak{F}_0(\mathfrak{X}), A \in \mathfrak{X}, (\Phi_0, A) \in q_{\tau_l}$ and $\varphi \in \mathfrak{F}_0(A)$. Then $\forall U \in \varphi \cap \tau : U^- \in \Phi_0$, implying that $\mathfrak{B} := \{B_{U,\mathfrak{A}} := \{\varphi_0 \in \mathfrak{F}_0(X) |\ \exists K \in \mathfrak{A} : \varphi_0 \in \mathfrak{F}_0(K \cap U)\} |\ \mathfrak{A} \in \Phi_0, U \in \varphi \cap \tau\}$ is a base for a filter, which refines $\Phi_0^{\uparrow}$, and there must be an ultrafilter Φ_1, containing this base, therefore containing $\Phi_0^{\uparrow}$, too. Now, obviously $U \in \bigcap_{\varphi \in B_{U,\mathfrak{A}}} \varphi$ holds for every $U \in \varphi \cap \tau$, implying $\varphi \cap \tau \subseteq \Phi_1^{\cup\cap}$.
Otherwise, let $\Phi_0 \in \mathfrak{F}_0(\mathfrak{X}), A \in \mathfrak{X}$ and $(\Phi_0, A) \notin q_{\tau_l}$. This means, $\exists O \in \tau : A \cap O \neq \emptyset \wedge O^{-\mathfrak{x}} \notin \Phi_0$. Especially, there exists an element $a \in A \cap O$ and so O is an open neighbourhood of a. Now, Φ_0 is an ultrafilter on $\mathfrak{X}$, so $O^{-\mathfrak{x}} \notin \Phi_0$ just implies $O^{+\mathfrak{x}} \in \Phi_0$, leading to $\mathfrak{F}_0(O^c) \in \Phi_0^{\uparrow}$, yielding $\forall \Phi_1 \in \mathfrak{F}_0(\Phi_0^{\uparrow}) : \mathfrak{F}_0(O^c) \in \Phi_1$. But then

we have $\forall \Phi_1 \in \mathfrak{F}_0(\Phi_0^{\uparrow}) : O^c \in \Phi_1^{\cup\cap}$ and therefore $O \notin \Phi_1^{\cup\cap}$. So, none of these $\Phi_1^{\cup\cap}$ contains $\dot{a} \cap \tau$. ■

16 Corollary
Let (X, τ) be a topological space, $\mathfrak{X} \subseteq \mathfrak{P}_0(X)$, $\Phi \in \mathfrak{F}_0(\mathfrak{X})$ and let A be a compact subset of X. Then Φ converges to A w.r.t. the Vietoris-topology, iff

(1) $\forall \Phi_1 \in \mathfrak{F}_0(\Phi^{\uparrow}) : \exists a \in A : (\Phi_1^{\cup\cap}, a) \in q_\tau$ *and*

(2) $\forall a \in A : \exists \Phi_1 \in \mathfrak{F}_0(\Phi_0^{\uparrow}) : (\Phi_1^{\cup\cap}, a) \in q_\tau$.

Proof: Let $\Phi \in \mathfrak{F}_0(\mathfrak{X})$ converge to A w.r.t. the Vietoris-topology. Because A is compact, every ultrafilter φ on A converges on A, i.e. it contains all open neighbourhoods of a point $a \in A$. But then $\Phi_1^{\cup\cap} \supseteq \varphi \cap \tau$ contains them, too. So, (1) follows from Proposition 14, and (2) follows from the fact, that $\dot{a}$ itself is an ultrafilter, together with proposition 15.
If otherwise Φ doesn't converge to A w.r.t. the Vietoris-topology, then it doesn't converge w.r.t. τ_l or w.r.t. τ_V^+. Then the second parts of the proofs of propositions 15 or 14, respectively, provide that (2) or (1), respectively, doesn't hold. ■

Now, we will go on to define convergences on the set of all filters on a topological space just by applying the requirements above to this case:

17 Definition
Let (X, τ) be a topological space and $\mathfrak{X} \subseteq \mathfrak{F}(X)$, then pseudotopological convergences $q_l(\tau)$ and $q_u(\tau)$ on $\mathfrak{X}$ are defined by

$$(\Psi, \psi) \in q_l(\tau) \quad :\Leftrightarrow \quad \forall \varphi \in \mathfrak{F}_0(\psi) : \exists \Phi_1 \in \mathfrak{F}_0(\Psi^{\uparrow}) : \Phi_1^{\cup\cap} \supseteq \varphi \cap \tau \ , \qquad (1)$$
$$(\Psi, \psi) \in q_u(\tau) \quad :\Leftrightarrow \quad \forall \Phi_1 \in \mathfrak{F}_0(\Psi^{\uparrow}) : \exists \varphi \in \mathfrak{F}_0(\psi) : \Phi_1^{\cup\cap} \supseteq \varphi \cap \tau \qquad (2)$$

for ultrafilters Ψ *on $\mathfrak{X}$ and filters $\psi \in \mathfrak{X}$, together with the "pseudotopological convention", that a filter on $\mathfrak{X}$ converges to an element of $\mathfrak{X}$, iff every refining ultrafilter does.*
A third convergence $q_V(\tau)$ is defined just by

$$(\Phi, \varphi) \in q_V(\tau) :\Leftrightarrow \forall \Phi_0 \in \mathfrak{F}_0(\Phi) : (\Phi_0, \varphi) \in q_l(\tau) \wedge (\Phi_0, \varphi) \in q_u(\tau) \ ,$$

and we call it the **strong Vietoris-pseudotopology** *on $\mathfrak{F}(X)$.*

In order to verify, that this defines indeed a pseudotopological convergence on $\mathfrak{F}(X)$, we have at first to remember, that our defining requirements only apply to ultrafilters and then the *generated* pseudotopology will be taken. So, it remains only to verify, that all singleton filters converge to their generating singleton - but this is very easy to see.
Although this convergence is quite strong, we will get a compactness result for this.

For further investigations, our interest will focus a somewhat weaker, but quite similar convergence, defined here not for arbitrary filters, but for the compactoid ones.

18 Definition
Let (X,τ) be a topological space, then convergences $q'_l(\tau)$ and $q'_u(\tau)$ on $\mathfrak{C}(X)$ are defined by

$$\begin{array}{rcl}(\Psi,\psi)\in q'_l(\tau) & :\Leftrightarrow & \forall\varphi\in\mathfrak{F}_0(\psi):\exists\Phi_1\in\mathfrak{F}_0(\Psi^{\uparrow}):\forall A\in\psi:A\cap q_\tau(\Phi_1^{\sqcup\sqcap}\cap\varphi)\neq\emptyset\ ,\\ (\Psi,\psi)\in q'_u(\tau) & :\Leftrightarrow & \forall\Phi_1\in\mathfrak{F}_0(\Psi^{\uparrow}):\exists\varphi\in\mathfrak{F}_0(\psi):\forall A\in\psi:A\cap q_\tau(\Phi_1^{\sqcup\sqcap}\cap\varphi)\neq\emptyset\end{array}$$

for ultrafilters Ψ *on* $\mathfrak{F}(X)$*, together with the "pseudotopological convention", that a filter on* $\mathfrak{F}(X)$ *converges to a filter on* X*, iff every refining ultrafilter does.*
A third convergence $q'_V(\tau)$ is defined just by

$$(\Phi,\varphi)\in q'_V(\tau):\Leftrightarrow\forall\Phi_0\in\mathfrak{F}_0(\Phi):(\Phi_0,\varphi)\in q'_l(\tau)\wedge(\Phi_0,\varphi)\in q'_u(\tau)\ ,$$

and we call it the **Vietoris-pseudotopology** *on* $\mathfrak{C}(X)$.

To check, that this really defines a pseudotopology is easy again by the same reasons as above.

19 Proposition
Let (X,τ) be a topological space. Then on the set $\mathfrak{C}(X)$ of all compactoid filters on X hold

$$\begin{array}{rcl}q_l(\tau) & \subseteq & q'_l(\tau)\ ,\\ q_u(\tau) & \subseteq & q'_u(\tau)\ ,\ \textit{and consequently}\\ q_V(\tau) & \subseteq & q'_V(\tau)\ .\end{array}$$

Proof: If Ψ,ψ fulfill the requirements to converge in one of the senses of definition 17, the corresponding requirement of definition 18 is fulfilled with the same Φ_1 respectively φ_0. We have just to observe, that a filter, which contains all open members of a convergent filter, converges at least to the same points. ■

From this and from corollary 16 we see, that $q'_V(\tau)$ coincides with the Vietoris-topology on $K(X)$, provided, we identify the compact sets with their generated principal filters.

20 Lemma
Let (X,τ) be a compact topological space. Then $(\mathfrak{C}(X),q_V(\tau))$ is compact, too.

Proof: Let $\Phi\in\mathfrak{F}_0(\mathfrak{C}(X))$. We will show, that Φ converges in $q_V(\tau)$ to the filter

$$\varphi_\Phi:=\left\langle\left\{adh(\bigcap_{\chi\in\mathfrak{A}}\chi)\ \middle|\ \mathfrak{A}\in\Phi\right\}\right\rangle\ ,$$

which is compactly generated, and thus compactoid, because (X,τ) is compact, and so the (by proposition 6 closed) generating sets are compact, too.
To prove $(\Phi,\varphi_\Phi)\in \underline{q_l(\tau)}$, let $\varphi_0\in\mathfrak{F}_0(\varphi_\Phi)$ be given. Then we have for all $U\in\varphi_0\cap\tau$, that $\forall\mathfrak{A}\in\Phi: U\cap\overline{\bigcup_{\chi\in\mathfrak{A}} adh(\chi)}\neq\emptyset$ and therefore $U\cap\bigcup_{\chi\in\mathfrak{A}} adh(\chi)\neq\emptyset$, because of the closedness-properties and the fact, that U is open. Thus, for all $U\in\varphi_0\cap\tau$ and all $\mathfrak{A}\in\Phi$, the set

$$M_{U,\mathfrak{A}} := \{\psi\in\mathfrak{F}_0(X)|\ \exists\chi\in\mathfrak{A}, u\in U:\psi\supseteq\chi\wedge(\psi,u)\in q_\tau\}$$

is not empty. Obviously, for $U_1,U_2\in\varphi_0\cap\tau$ and $\mathfrak{A}_1,\mathfrak{A}_2\in\Phi$ we get $M_{U_1\cap U_2,\mathfrak{A}_1\cap\mathfrak{A}_2}\subseteq M_{U_1,\mathfrak{A}_1}\cap M_{U_2\cap\mathfrak{A}_2}$, so $\mathfrak{M}:=\{M_{U,\mathfrak{A}}|\ U\in\varphi_0\cap\tau,\mathfrak{A}\in\Phi\}$ is a filterbase, and there exists an ultrafilter Φ_1, which contains $\mathfrak{M}$. Observe now, that $M_{U,\mathfrak{A}}\in\Phi_1$ for all $\mathfrak{A}\in\Phi, U\in\varphi_0\cap\tau$ and $M_{U,\mathfrak{A}}\subseteq\bigcup_{\chi\in\mathfrak{A}}\mathfrak{F}_0(\chi)\subseteq\mathfrak{F}_0(\bigcap_{\chi\in\mathfrak{A}}\chi)$ together imply $\Phi_1\supseteq\Phi^\uparrow$. Furthermore, every $\psi\in M_{U,\mathfrak{A}}$ converges to an element of U, so it must contain the open set U, yielding $U\in\bigcup_{\psi\in M_{U,\mathfrak{A}}}\psi$. Now, all $M_{U,\mathfrak{A}}$ with $U\in\varphi\cap\tau$ and $\mathfrak{A}\in\Phi$ belong to Φ_1, which implies $\Phi_1^{\cup\cap}\supseteq\varphi_0\cap\tau$. So, the defining requirement for $q_l(\tau)$ in 17(1) is fulfilled.
To prove, $(\Phi,\varphi_\Phi)\in q_u(\tau)$, let $\Phi_1\in\mathfrak{F}_0(\Phi^\uparrow)$ be given. Because (X,τ) is compact, every ultrafilter on X converges w.r.t. q_τ, i.e. $\forall\psi\in\mathfrak{F}_0(X): q_\tau(\psi)\neq\emptyset$. So, there exists a map $\lambda:\mathfrak{F}_0(X)\to X$ with $\forall\psi\in\mathfrak{F}_0(X):\lambda(\psi)\in q_\tau(\psi)$. Now, $\varphi_0:=\lambda(\Phi_1)$ is an ultrafilter on X, because Φ_1 is an ultrafilter on $\mathfrak{F}_0(X)$. Moreover, $\varphi_0\supseteq\varphi_\Phi$ holds, because $\forall T\in\varphi_\Phi:\exists\mathfrak{A}\in\Phi: T\supseteq adh(\bigcap_{\chi\in\mathfrak{A}}\chi)=q_\tau(\mathfrak{F}_0(\bigcap_{\chi\in\mathfrak{A}}\chi))\supseteq\lambda(\mathfrak{F}_0(\bigcap_{\chi\in\mathfrak{A}}\chi))\in\lambda(\Phi^\uparrow)\subseteq\lambda(\Phi_1)$. Now, let $U\in\varphi_0\cap\tau$. Then there is an $M\in\Phi_1$, s.t. $\lambda(M)\subseteq U$, i.e. all elements of M converge to elements of U, so they all must contain the open neighbourhood U. But then $U\in\bigcup_{\psi\in M}\psi$ holds. This is valid for all $U\in\varphi_0\cap\tau$, yielding $\varphi_0\cap\tau\subseteq\Phi_1^{\cup\cap}$; so the defining requirement for $q_u(\tau)$ in 17(2) is fulfilled. ■

21 Corollary
Let (X,τ) be a compact topological space. Then $(\mathfrak{C}(X), q'_V(\tau))$ is compact, too.

Proof: Follows directly from lemma 20 and proposition 19. ■

5 The Mizokami-map

We will map the function space $(C(X,Y),q_c)$ into the function space $(C(\mathfrak{C}(X),\mathfrak{C}(Y)),\ q_p)$, where $\mathfrak{C}(X),\mathfrak{C}(Y)$ are endowed with the Vietoris-pseudotopologies by

$$\mu: C(X,Y)\to\mathfrak{C}(Y)^{\mathfrak{C}(X)}: f\to\mu(f):\mu(f)(\varphi):=f(\varphi)$$

Here, proposition 9 ensures, that we really map into $\mathfrak{C}(Y)^{\mathfrak{C}(X)}$, not only into $\mathfrak{F}(Y)^{\mathfrak{C}(X)}$. We call this mapping μ the *Mizokami-map*.

22 Proposition
Let $(X,\tau),(Y,\sigma)$ be topological spaces. Then for the map

$$\mu : C(X,Y) \to \mathfrak{C}(Y)^{\mathfrak{C}(X)} : f \to \mu(f) : \mu(f)(\varphi) := f(\varphi)$$

hold

(1) μ *is injective,*

(2) $\forall \mathcal{F} \in \mathfrak{F}(C(X,Y)), \varphi \in \mathfrak{C}(X) : (\mu(\mathcal{F})(\varphi))^{\cup\cap} \supseteq \mathcal{F}(\varphi)$ *and*

(3) $\forall \Phi \in \mathfrak{F}(\mathfrak{C}(X)), f \in C(X,Y) : f(\Phi^{\uparrow}) = (\mu(f)(\Phi))^{\uparrow}$.

Proof: (1) follows directly from the fact, that the singleton-filters are compactoid. So, if $\mu(f) = \mu(g)$, especially $\forall x \in X : \mu(f)(\overset{\bullet}{x}) = \mu(g)(\overset{\bullet}{x})$ and therefore $\forall x \in X : f(x) = g(x)$ holds.
(2): $M \in \mathcal{F}(\varphi) \Leftrightarrow \exists F \in \mathcal{F}, P \in \varphi : \forall g \in F : g(P) \subseteq M \Rightarrow \exists F \in \mathcal{F} : \forall g \in F : \exists P_g \in \varphi : g(P_g) \subseteq M \Leftrightarrow \exists F \in \mathcal{F} : M \in \bigcup_{g \in F} g(\varphi) \Leftrightarrow M \in (\mu(\mathcal{F})(\varphi))^{\cup\cap}$.
(3): We have

$$\begin{aligned}
\mathfrak{M} \in f(\Phi^{\uparrow}) \quad \Leftrightarrow \exists \mathfrak{A} \in \Phi : \quad & f(\mathfrak{F}_0(\bigcap_{\chi \in \mathfrak{A}} \chi)) \subseteq \mathfrak{M} \\
& \Leftrightarrow \mathfrak{F}_0(f(\bigcap_{\chi \in \mathfrak{A}} \chi)) \subseteq \mathfrak{M} \text{ (by proposition 4)} \\
& \Leftrightarrow \mathfrak{F}_0(\bigcap_{\chi \in \mathfrak{A}} f(\chi)) \subseteq \mathfrak{M} \text{ (by proposition 1)} \\
& \Leftrightarrow \mathfrak{M} \in (\mu(f)(\Phi))^{\uparrow} \ \blacksquare
\end{aligned}$$

23 Lemma
Let $(X,\tau),(Y,\sigma)$ be topological spaces. Then with the map

$$\mu : C(X,Y) \to \mathfrak{C}(Y)^{\mathfrak{C}(X)} : f \to \mu(f) : \mu(f)(\varphi) := f(\varphi)$$

holds, that $\mu(f)$ is continuous w.r.t. $q'_V(\tau), q'_V(\sigma)$ for all $f \in C(X,Y)$.

Proof: Let $f \in C(X,Y)$ and $\Phi \in \mathfrak{F}_0(\mathfrak{C}(X)), \varphi \in \mathfrak{C}(X)$ with $(\Phi,\varphi) \in q'_V(\tau)$ be given. For every $\psi_0 \in \mathfrak{F}_0(f(\varphi))$ there is a $\varphi_0 \in \mathfrak{F}_0(\varphi)$ with $f(\varphi_0) = \psi_0$, by corollary 3. Because Φ converges to φ w.r.t. $q'_l(\tau)$, we know, that there is a $\Phi_1 \in \mathfrak{F}_0(\Phi^{\uparrow})$ such that every member A of φ contains an element a, s.t. φ_0 and $\Phi_1^{\cup\cap}$ both converge to a w.r.t. τ. Thus every member of $f(\varphi)$ contains an element $f(a)$ s.t. $\psi = f(\varphi_0)$ and $f(\Phi_1^{\cup\cap})$ both converge to $f(a)$, because of the continuity of f. But then $f(\Phi_1)^{\cup\cap}$ converges to $f(a)$ because of proposition 13(2), and we know $f(\Phi_1) \in f(\mathfrak{F}_0(\Phi^{\uparrow})) = \mathfrak{F}_0(f(\Phi^{\uparrow})) = \mathfrak{F}_0(\mu(f)(\Phi)^{\uparrow})$ from the propositions 4 and 22(3). So we find $(\mu(f)(\Phi), \mu(f)(\varphi)) \in q'_l(\sigma)$.
Furthermore, for every $\Phi_2 \in \mathfrak{F}_0(\mu(f)(\Phi)^{\uparrow})$ we observe $\mathfrak{F}_0(\mu(f)(\Phi)^{\uparrow}) = \mathfrak{F}_0(f(\Phi^{\uparrow})) =$

$f(\mathfrak{F}_0(\Phi^{\uparrow}))$ because of the propositions 22(3) and 4, and conclude, that there exists $\Phi_1 \in \mathfrak{F}_0(\Phi^{\uparrow})$ with $f(\Phi_1) = \Phi_2$. Now, Φ converges to φ w.r.t. $q'_u(\tau)$, so there exists $\varphi_0 \in \mathfrak{F}_0(\varphi)$ s.t. every $A \in \varphi$ contains an element a, to which φ_0 and $\Phi_1^{\Cup\Cap}$ both converge w.r.t. τ. Thus every $f(A) \in f(\varphi)$ contains an element $f(a)$ s.t. $f(\Phi_1^{\Cup\Cap})$ and $f(\varphi_0) \in \mathfrak{F}_0(f(\varphi))$ both converge to $f(a)$ w.r.t. σ, because of the continuity of f. Now, from proposition 13(2) it follows, that $f(\Phi_1)^{\Cup\Cap} = \Phi_2^{\Cup\Cap}$ converges to $f(a)$, too. So we find $(\mu(f)(\Phi), \mu(f)(\varphi)) \in q'_u(\sigma)$, implying now $(\mu(f)(\Phi), \mu(f)(\varphi)) \in q'_V(\sigma)$ because of the above proven q'_l-convergence, and therefore, because this holds for all $(\Phi, \varphi) \in q'_V(\tau)$, the continuity of $\mu(f)$ follows. ∎

24 Lemma
Let $(X,\tau), (Y,\sigma)$ be topological spaces. Then the map

$$\mu : C(X,Y) \to C(\mathfrak{C}(X), \mathfrak{C}(Y)) : f \to \mu(f) : \varphi \to f(\varphi)$$

is continuous and injective, where $C(X,Y)$ is endowed with the structure q_c of continuous convergence, $C(\mathfrak{C}(X), \mathfrak{C}(Y))$ with the structure q_p of pointwise convergence, for $\mathfrak{C}(X)$ and $\mathfrak{C}(Y)$ being equipped with the Vietoris-pseudotopologies $q'_V(\tau)$ and $q'_V(\sigma)$, respectively.

Proof: By proposition 22 we know, that μ is injective and lemma 23 says, that μ maps $C(X,Y)$ into $C(\mathfrak{C}(X), \mathfrak{C}(Y))$. To prove continuity of μ, let $\mathcal{F} \in \mathfrak{F}(C(X,Y)), f \in C(X,Y)$ with $(\mathcal{F}, f) \in q_c$ and an arbitrary $\varphi \in \mathfrak{C}(X)$ be given. Then for all $\psi_0 \in \mathfrak{F}_0(f(\varphi))$, by corollary 3 there exists a $\varphi_0 \in \mathfrak{F}_0(\varphi)$ such that $f(\varphi_0) = \psi_0$. Now, we have naturally $\forall B \in f(\varphi) : \exists A_B \in \varphi : f(A_B) \subseteq B$, and because of the compactoidness of φ we know $\exists a \in A_B : (\varphi_0, a) \in q_\tau$. By the continuity of f we get now $(f(\varphi_0), f(a)) \in q_\sigma$, i.e. $(\psi_0, f(a)) \in q_\sigma$. Because of the continuous convergence of $\mathcal{F}$ to f, we find $(\mathcal{F}(\varphi_0), f(a)) \in q_\sigma$.
Observe now, that $\mu(\mathcal{F})(\varphi_0)$ is a filter on $\mathfrak{F}_0(Y)$, which refines $\mu(\mathcal{F})(\varphi)^{\uparrow}$, because φ_0 is an ultrafilter and it refines φ. This yields $\mathfrak{F}_0(\mu(\mathcal{F})(\varphi_0)) \subseteq \mathfrak{F}_0(\mu(\mathcal{F})(\varphi)^{\uparrow})$.
So, let $\Phi_1 \in \mathfrak{F}_0(\mu(\mathcal{F})(\varphi_0))$. Then $\Phi_1^{\Cup\Cap} \supseteq (\mu(\mathcal{F})(\varphi_0))^{\Cup\Cap}$, implying $\Phi_1^{\Cup\Cap} \supseteq \mathcal{F}(\varphi_0)$ by proposition 22(2), thus $\Phi_1^{\Cup\Cap}$ converges to $f(a)$, because $\mathcal{F}(\varphi_0)$ does. All in all, every $B \in f(\varphi)$ contains an element $b = f(a)$ to which both, ψ_0 and $\Phi_1^{\Cup\Cap}$, converge. This holds for all $\psi \in \mathfrak{F}_0(\varphi)$, implying $(\mu(\mathcal{F})(\varphi), \mu(f)(\varphi)) \in q'_l(\sigma)$.
Furthermore, let $\Phi_1 \in \mathfrak{F}_0(\mu(\mathcal{F})(\varphi)^{\uparrow})$. Then $\Phi_1^{\Cup\Cap}$ is an ultrafilter on Y by proposition 13. Thus, the collection $\mathfrak{B} := \{O \in \sigma | \ O \notin \Phi_1^{\Cup\Cap}\}$ is closed under finite unions. Assume now, that every refining ultrafilter of $f(\varphi)$ would contain an element of $\mathfrak{B}$. Then by lemma 5, the filter $f(\varphi)$ itself must contain an open set, which doesn't belong to $\Phi_1^{\Cup\Cap}$. But proposition 22(2) ensures $\Phi_1^{\Cup\Cap} \supseteq \mathcal{F}(\varphi)$ and by lemma 12 we know, that $\mathcal{F}$ converges $\mathfrak{C}(X)$-continuously to f, just yielding $\mathcal{F}(\varphi) \supseteq f(\varphi) \cap \sigma$ - a contradiction. Thus, there must exist an refining ultrafilter ψ of $f(\varphi)$, whose open members are all contained in $\Phi_1^{\Cup\Cap}$, too, so $\Phi_1^{\Cup\Cap}$ converges to the same points as ψ does, and consequently $(\mu(\mathcal{F})(\varphi), \mu(f)(\varphi)) \in q'_u(\sigma)$ holds, yielding $(\mu(\mathcal{F})(\varphi), \mu(f)(\varphi)) \in q'_V(\sigma)$,

because of the result above. These convergence relations are valid for all $\varphi \in \mathfrak{C}(X)$, so $(\mu(\mathcal{F}), \mu(f)) \in q_p$ follows. ■

25 Theorem
Let $(X, \tau), (Y, \sigma)$ be topological spaces and $\mathcal{H}$ an evenly continuous subset of $C(X, Y)$. Then the map

$$\mu : \mathcal{H} \to C(\mathfrak{C}(X), \mathfrak{C}(Y)) : f \to \mu(f) : \varphi \to f(\varphi)$$

is continuous, injective and its inverse map from $\mu(\mathcal{H})$ to $\mathcal{H}$ is continuous, too, where $\mathcal{H}$ is endowed with the structure q_c of continuous convergence, $C(\mathfrak{C}(X), \mathfrak{C}(Y))$ with the structure q_p of pointwise convergence, for $\mathfrak{C}(X)$ and $\mathfrak{C}(Y)$ being equipped with the Vietoris-pseudotopologies $q'_V(\tau)$ and $q'_V(\sigma)$, respectively.

Proof: According to lemma 24, we have only to show, that the inverse map is continuous. So, let $\mathcal{F} \in \mathfrak{F}_0(\mathcal{H})$ with $\mu(\mathcal{F}) \xrightarrow{p} \mu(f) \in \mu(\mathcal{H})$ be given.
Because all singleton-filter $\dot{x}, x \in X$ are compactoid, we have at first $\forall x \in X$: $\mu(\mathcal{F})(\dot{x}) \xrightarrow{q'_V(\sigma)} \mu(f)(\dot{x}) = f(\dot{x})$, thus from the definition of q'_V we get $\forall \Psi \in \mathfrak{F}_0(\mu(\mathcal{F})(\dot{x})^{\uparrow}) : \forall A \in f(\dot{x}) : A \cap q_\sigma(\Psi^{\sqcup\sqcap}) \neq \emptyset$. Observe now, that $\mu(\mathcal{F})(\dot{x})$ is itself an ultrafilter on $\mathfrak{F}_0(Y)$ finer than $\mu(\mathcal{F})(\dot{x})^{\uparrow}$, because $\mathcal{F}$ is an ultrafilter and for each $F \in \mathcal{F}$, all singleton filters $\dot{g(x)}, g \in F$, belong to $\mathfrak{F}_0(\bigcup_{g \in F} \mu(g)(\dot{x}))$. Taking $\{f(x)\}$ for A, we get then $\mu(\mathcal{F})(\dot{x})^{\sqcup\sqcap} \xrightarrow{\sigma} f(x)$ from the above. But it is easy to see, that $\mu(\mathcal{F})(\dot{x})^{\sqcup\sqcap} = \mathcal{F}(x)$, so $\mathcal{F}(x)$ converges to $f(x)$ for all $x \in X$ and consequently, $\mathcal{F}$ converges pointwise to f. Now, from the even continuity of $\mathcal{H}$ follows $(\mathcal{F}, f) \in q_c$. ■

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