FOUNDATIONS OF TOPOLOGY
BASIC CONCEPTS OF SET THEORY

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CONTENTS

1 INTRODUCTION

The barber is a man in town who shaves all those,
and only those, men in town who do not shave themselves.
Who shaves the barber?

The "barber paradoxon" is a very interesting question that we will be able to answer at the end of section three. The need to solve problems like this led to create different set theories. We will focus on two of them, namely the Neumann- Bernays- Gödel and the Zermelo- Fraenkel set theory, which are basically equivalent for the most part. But first we have to define some fundamental terms.

2 SETS, CLASSES, CONGLOMERATES

2.1 Sets

We want to assume that the following defnitions are well-known:
property, subset, empty set, power set, union, intersection, relative complement, disjoint union

DEFINITION.

- Let x and y be elements. $(x, y) := \{\{x\}, \{x, y\}\}$ is defined as **ordered pair**.
- Let X and Y be sets. $X \times Y := \{(x, y) \mid x \in X \wedge y \in Y\}$ is defined as **Cartesian Product**.

We can form the following well-known sets:
$\mathbb{N}$: all natural numbers,
$\mathbb{Z}$: all integers,
$\mathbb{Q}$: all rational numbers,
$\mathbb{R}$: all real numbers,
$\mathbb{C}$: all complex numbers

2.2 Classes

Dealing with "large collections of sets", there is the concept of "classes"
In particular, we require that

- the member of each class are sets
- for every property P one can form the **class of all sets with property** P.

DEFINITION.

- The largest class, i.e. the class of all sets, is called the **universe**, denoted by $\mathcal{U}$.

Hence, classes are the subcollections of $\mathcal{U}$, and every set is a class.

- Classes that are not sets are called **proper classes**. They cannot be members of any class.
- Sets are also called **small classes**, and proper classes are also called **large classes**.

The framework of sets and classes suffices for defining such entities as the category of sets, of vector spaces, of functors between these categories, and natural transformations between functors. But when we try to perform certain constructions with categories the framework is limited, i.e. we need a further level of generality:

2.3 Conglomerates

The concept of "conglomerate" has been created to deal with "collections of classes". We require the following:

- Every class is a conglomerate.
- For every "property" P, one can form the conglomerate of all classes with property P.
- Conglomerates are closed under analogues of the usual set-theoretic constructions, i.e. they are closed under the formation of pairs, unions, products (of conglomerate-indexed families), etc.

Consequently, there is the following **hierarchy of "collections"** :

- **conglomerates**
  - **classes** = subcollections of the universe $\mathcal{U}$
    - **sets** = small classes = elements of $\mathcal{U}$

3 AXIOMATIC SYSTEMS

In the next section we concentrate on two set theories, namely *the von Neumann- Bernays-Gödel* (NBG) set theory and the *Zermelo-Fraenkel* one, containing the axiom of choice (ZFC). The Zermelo- Fraenkel set theory is preferred for plain set theoretical arguments and the Neumann- Bernays- Gödel one is easier to use when you deal with problems of practical application. The ambition of the theories, that are based on different axioms, is not to find an explanation for what is a set, but to define axiomatic properties for exactly such sets.

3.1 The concept of "set"

The father of set theory, Cantor, defined the term "set" as a

> *collection* M *into a whole of definite and separate objects of our intuition or thought. These objects are called the "elements" of* M.

This is a less "mathematical" definition, for example consider the expression "collection into a whole". You can see that even Cantor missed words that could describe the complex term "set" perfectly. To make the statement more precise we could write it in this way: "For every property $P(x)$ exists the set M of all objects x with property $P(x) : M = \{x : P(x)\}$". These "properties" were eventually adapted by Zermelo and Fraenkel, keeping Cantor's definition in mind, and led to the formation of the class comprehension axiom (Axiom 2).
So here is the modern definition for this fundamental term.

DEFINITION. The class A is called a set, if and only if there exists a class $\mathcal{A}$ with $A \in \mathcal{A}$.

So sets are exactly those classes, that are also element of another class. You could say that sets are a special form of classes. On the other side, classes that are not element of any other class, thus no set, are called "unproper classes" [Unmengen].
Therefore we can follow that sets and classes are equal, as far as dimensions are compatible (i.e. the same elements), which also leads to the extensionality axiom (Axiom 1). Furthermore we can conclude that every set is a class, but not every class is a set, only classes that are "not too large" are sets.

3.2 Von Neumann–Bernays–Gödel set theory (NBG)

The Neumann- Bernays- Gödel (NBG) set theory defines axioms for set theory that are based on the Zermelo- Fraenkel (ZFC) version, which we come to know later. In 1925 John von Neumann was not very pleased with the set theory version Zermelo and Fraenkel created earlier. He compensated the weakness of ZFC's *axiom of replacement* and created a new one that contained the term of "function", so that he ended up with a finite system of 23 axioms. In 1937 Paul Bernays captured Neumann's new idea of set theory and integrated the strict separation of classes and sets although he was not content with his own final version. Kurt Gödel then further simplified the theory in his "continuum hypothesis" in 1940 which has laid the foundation for our modern conception of NBG set theory.

3.2.1 *Axioms*

Before we start bothering about the axioms, we need to define some fundamental terms. First we need the term of "class" which was already explained earlier. Then we need to define the operator $\in$ which means "element of" and suppose that all elements of sets are also sets. Furthermore we have to assume that for two classes A and B the statement $A \in B$ is either true or false. If it is false we can also write $A \notin B$.

DEFINITION. A class A is called subclass of class B (short: $A \subseteq B$) if and only if, the statement $\forall x : x \in A \Rightarrow x \in B$ holds.
The classes A,B are called equal (short: $A = B$) if and only if, the statements $A \subseteq B$ and $B \subseteq A$ hold.

Axiom 1. (Extensionality)
Let A be a class. $(x \in A) \wedge (y = x) \Rightarrow y \in A$.

By "class equality" we were already allowed to substitute the class variable on the right hand side of the element relation. Now the first axiom allows us to substitute the class variable on the left hand side of the relation, if the variables denote the same class.

Axiom 2. (Class comprehension)
For any formula $p(x)$, containing no quantifiers over classes and no class variable A, there exists a class A, which elements are exactly the ones that are contained in the set $\{x\}$, for that $p(x)$ is true, so

$$(x \in A) \Rightarrow (x \quad \text{is set}) \wedge p(x).$$

So now we know the solution to our problem in the very beginning. The class of sets, that do not have themselves as an element, are no sets. This insight tells us the answer: the barber is not a man. Additionally this axiom allows us three more operations on classses:

- union: $\mathcal{A} \cup \mathcal{B}$
- intersection: $\mathcal{A} \cap \mathcal{B}$
- Cartesian product: $\mathcal{A} \times \mathcal{B}$
- new terms for classes like relations and mappings

Axiom 3. (Empty set)
$\emptyset$ is a set.

By **Axiom 2** the empty set can also be written as $\emptyset := \{x | \ (x \quad \text{is set}) \wedge (x \neq x)\}$ and hence is subclass of every class. It also ensures that at least one class is actually a set.

The following axioms state that certain constructions of sets are again sets.

Axiom 4. (Pairing) If A and B are different sets, then $M := \{x| \quad (x = A) \vee (x = B)$ is a set. It can also be written as $\{A, B\}$.

Axiom 5. (Union) If $\mathscr{A}$ is a set, then

$$\bigcup_{A \in \mathscr{A}} A := \{x| \quad \exists A \in \mathscr{A} : x \in A\}$$

is a set.

Axiom 6. (Axiom of replacement)
If A is a set and $f : A \to \mathscr{A}$ is a map, then $f(A)$ is also a set.

Axiom 7. (Intersection)
If A is a set, then for any class $\mathscr{A}$, $A \cap \mathscr{A}$ is also a set.

Out of **Axiom 2** we can form the class $\{x|x \in A \wedge p(x)\}$ which is now we know a set. We do not need to write "x is set" anymore because of the statement $x \in A$ by **Axiom 2**.
We can conclude that this is also a set:

$$\bigcap_{A \in \mathscr{A}} A := \{x \in S | \forall A \in \mathscr{A} : x \in A\}$$

> **Axiom 8.** (Power set)
> If A is a set, then the power set $\mathfrak{P} := \{M|M \quad \text{is set} \;\wedge\; M \subseteq A\}$ is a also a set.

We can now construct a class such as $\mathscr{P}(A) := \{\mathcal{B}| \; \mathcal{B} \text{ is a set} \wedge \mathcal{B} \subseteq \mathscr{A}\}$ which we call *power class* of A.

> **Axiom 9.** (Atoms)
> Every non-empty set A contains an element $a \in A$ with $a \cap A = \emptyset$.

In other words the set A contains elements that are not again made up of elements of A, i.e. the set consists of so called "atoms".

Proposition 1. 1. No set is element of itself.

2. For two sets A,B the statements $A \in B$ and $B \in A$ never hold at the same time.

Proof. 1. The statement is obvious for the empty set. Let A be non-empty with $A \in A$. By axiom 8 $\{A\}$ is also a set. Because A is element of A, A has the single element A. It follows that A does not contain any atoms which leads to a contradiction.

2. The statement is again obvious for the empty set. By axiom 4 $\{A, B\}$ is also a set that would not have any atoms just like in (1.).

□

> **Axiom 10.** (Infinity)
> There exits a set A with following properties:
>
> - $\emptyset \in A$
> - if a is a member of A, then $a \cup \{a\}$ is also a member of A.

We can follow that the class of natural numbers $\mathbb{N}$ (by Peano) is a set:
Let $\mathcal{B} := \{B \in \mathscr{P}(A) | \; B \quad \text{has same properties as in axiom 10}\}$ and $N := \bigcap_{B \in \mathcal{B}} B$. The class N is also a set by axiom 5 and has the same properties as in axiom 10.
Let $x \cup \{x\}$ be the successor of $x \forall x \in N$. The *Peano axioms* are fulfilled in the way that $\emptyset \in N$ is "0", $\{\emptyset\}$ is "1", $\{\emptyset, \{\emptyset\}\}$ is "2", and so on.

> **Axiom 11.** (Axiom of choice)
> For every non-empty set $\mathscr{A}$ of non-empty sets, there exists a function
> $f : \mathscr{A} \to \bigcup_{A \in \mathscr{A}} A$, for which holds $\forall A \in \mathscr{A} : f(A) \in A$.

In other words: There exists a function, which can choose an element out of each set $A \in A$.
Equivalence relations and various proofs will be discussed later on.

3.3 Zermelo–Fraenkel set theory

The Zermelo- Fraenkel set theory (ZFC) was created starting 1930 and is an one-sorted theory in first-order logic. It contains only sets and there are no elements of sets that are not themselves sets or classes.
The main differences between NBG and ZFC are that ZFC contains only sets and NBG's objects are classes. ZFC has infinitely many axioms in comparison to NBG which has finitely many axioms. A statement in ZFC is provable in NBG if and only if it is provable in ZFC.

3.3.1 *Disadvantages of ZFC*

- *theory of ordinals*
  Cantor's theory of ordinal numbers was not revisited by Zermelo in his set theory, but von Neumann later used it to define his well-ordered sets with the use of the $\in$-relation.

- *criteria identifying classes that are too large to be sets*
  Zermelo avoided the use of large classes in his axioms which led to paradoxons. Unlike von Neumann who defined the *axiom of limitation of size* which states that a class X is not a member of any class, if and only if X can be mapped onto the universal class $\mathcal{U}$, to avoid paradoxons. He concluded the axiom of replacement and separation and that $\mathcal{U}$ can be well- ordered.

- *finitely many axioms*
  Von Neumann condemned Zermelo's imprecise idea of "definite propositional function" and defined his term "function" which only needed finitely many axioms.

- *axiom of regularity*
  Zermelo did not exclude non- well- founded sets in his axioms, so von Neumann defined the axiom of regularity, which states that all sets are well- founded. He proved relative consistency of this axioms by his former axiom of limitation of size.

DEFINITION.

- A subset $R \subseteq X \times Y$ is called **relation** between X and Y.
  For $(x, y) \in R$ we also say "x is in relation to y".
- The **inverse relation** of R is defined as $R^{-}1 := \{(y, x) \in Y \times X \mid (x, y) \in R\}$.
- $f \subseteq X \times Y$ is called **function** or **transformation** from X to Y , iff
  for each $x \in X$ there is precisely one $y \in Y$ satisfying $(x, y) \in f$.
  Notations: $f : X \to Y$, where X is the **domain** and Y is the **range** of f. Instead of $(x, y) \in f$ we also write $y = f(x)$, where y is the **image** of x under f. The set of all functions from X to Y is denoted by Y^X.
- Let $f : X \to Y$ and $A \subseteq X$. The function $f : A \to Y$, where $f_{|A} = g$ is called **restriction** of f to A.
  Likewise, let $A \subseteq X$ and $g : A \to Y$. The function $f : X \to Y$, where $f_{|A} = g$, is called **continuation** of g to X.

DEFINITION.

- Let X be a set (or class). The function $f : I \to X$, where I is a set (or class), is called an **indexed family** of elements of X. I is called te **index set** (or **index class**). Instead of $f(i), i \in I$ we usually write x_i , where x_i is the element of the family with index i respectively. Instead of $f : I \to X$ we write $(x_i)_{i \in I}$.
  For $I = \mathbb{N}$ it is about a **sequence**.
- If $X = \mathcal{P}(Y)$, the family of elements of X is called **family of subsets of** Y or **indexed family of sets**. Each set $\mathcal{M}$ of subsets of a set Y can be seen as an indexed family of sets: Take $I = \mathcal{M}$ and $f := id_{\mathcal{M}}$.

Using these indexed families of sets, there are some useful notations for unions and intersections:
Let $(M_i)_{i \in I}$ be an indexed family of sets with corresponding function f . Then we can write

- $\bigcup_{i \in I} M_i$ for the union $\bigcup_{M \in f(I)} M$
- $\bigcap_{i \in I} M_i$ for the intersection $\bigcap_{M \in f(I)} M$.

Another important set-theoretical construction is the Cartesian product of arbitrary indexed families of sets:

DEFINITION.

- Let $(M_i)_{i \in I}$ be an indexed family of sets.

$$\prod_{i \in I} M_i := \{x : I \to \bigcup_{i \in I} M_i \mid \forall i \in I : x(i) \in M_i\}$$

is defined as the **Cartesian product** of all M_i.

Notations:
Instead of $x(i)$ we usually write x_i and instead of $x \in \prod_{i \in I} M_i$ we can also write $(x_i)_{i \in I}$ or (x_i).

DEFINITION. Let X be a set. A relation $R \subseteq X \times X$ is called

- **reflexive** $\Leftrightarrow \forall x \in X : (x, x) \in R$
- **irreflexive** $\Leftrightarrow \forall x \in X : (x, x) \notin R$
- **transitive** $\Leftrightarrow \forall x, y, z \in X : (x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$
- **symmetric** $\Leftrightarrow \forall x, y \in X : (x, y) \in R \Rightarrow (y, x) \in R$
- **asymmetric** $\Leftrightarrow \forall x, y \in X : (x, y) \in R \Rightarrow (y, x) \notin R$
- **antisymmetric** $\Leftrightarrow \forall x, y \in X : (x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$
- **linear** $\Leftrightarrow \forall x, y \in X : (x, y) \in R \vee (y, x) \in R$
- **connex** $\Leftrightarrow \forall (x, y) \in X : (x, y) \in R \vee (y, x) \in R \vee x = y$

DEFINITION.

- A relation $R \subseteq X \times X$ is called **equivalence relation** on X iff it is reflexive and symmetric and transitive.
- Let R be an equivalence relation on X and $x \in X$. Then the set

$$[x]_R := \{z \in X \mid (x, z) \in R\}$$

is the **equivalence class** of x w.r.t. R.
Each element $y \in [x]_R$ is a **representative** of this equivalence class.

Obviously, because of reflexivity, $\bigcup_{x \in X} [x]_R = X$ for each equivalence relation R on a set X. Moreover, for equivalence classes either $[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \emptyset$: Suppose $[x]_R$ and $[y]_R$ are not disjoint, then there exists an element $z \in X$ which is element of both, i.e $(x, z) \in R$ and $(y, z) \in R$. Because of symmetry, $(z, y) \in R$ and because of transitivity, $(x, y) \in R$ and moreover $\forall v \in [y]_R : (x, v) \in R$. This implies $[y]_R \subseteq [x]_R$. Because of symmetry, $(z, x) \in R$ and this implies $[x]_R \subseteq [y]_R$ analogously. It follows $[x]_R = [y]_R$.

Let M be the set of subsets of X, where its elements are pairwise disjoint and the union of all elements of M is X. Then M is called **partition** of X. The elements of M are called **components of the partition**. This implies that each equivalence relation yields a partition, where the components of the partition are exactly the equivalence classes and vice versa.

The set of equivalence classes of X w.r.t. R is called **quotient space** of X and is denoted X/R.

DEFINITION. Let X be a set.

- A relation $R \subseteq X \times X$ is called **(partial) order** iff it is reflexive and transitive and antisymmetric.
- If R is a (partial) order and linear, then it is called **total order** or **linear order**.
- Let $\preceq$ be a partial order (or total order) defined on X. Then $(X, \preceq)$ is called a **(partially) ordered set** (or **totally ordered set**).

The relation $\leqslant$ defined on $\mathbb{R}$ is a total order, but the relation $\subseteq$ defined on the power set $\mathfrak{P}(X)$ of a set X which concludes more than one element is just a (partial) order.

Let $X := \{a, b, c\}$, then $\mathfrak{P}(X) = \{\{\emptyset\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. $\{a\} \nsubseteq \{b\} \wedge \{b\} \nsubseteq \{a\}$ implies that $(\mathfrak{P}(X), \subseteq)$ is not totally ordered.
But the subset $X_1 := \{\{\emptyset\}, \{a\}, \{a, b\}, \{a, b, c\}\}$ is totally ordered. These subsets of a (partially) ordered set are called **chains** in $(X, \leqslant)$.

DEFINITION.

- Let $(X, \preceq)$ and $(X', \preceq')$ be (partially) ordered sets. Then the function $f : X \to X'$ is called **isotonic**, iff $x \preceq y$ implies $f(x) \preceq' f(y)$.
- Let $(X, \preceq)$ and $(X', \preceq')$ be (partial) ordered sets and $f : X \to X'$ be a bijection, s.t. f and f^{-1} are isotonic. Then f is called an **order isomorphism**.
  If there exists an order isomorphism between X and X' then $(X, \preceq)$ and $(X', \preceq)$ are called **order isomorphic**.
- Let $(X, \preceq)$ be a (partially) ordered set and let $Y \subseteq X$.
  - An element $a \in X$ is called **upper** (or **lower**) **bound** of Y, iff for all $x \in Y : x \preceq a$ (or $a \preceq x$).
  - The *least upper bound* of X is called **supremum** and the *greatest lower bound* is called **infimum**.
- Let $(X, \preceq)$ be a (partially) ordered set. An element $a \in X$ is called **maximal** (or **minimal**), iff $x \in X$ and $a \preceq x$ (or $x \preceq a$) implies $x = a$.
- A totally ordered set X is called **well-ordered** (corrsponding order is called **well-order**), iff every non-empty subset of X has a least element in this ordering.

In a totally ordered set there is at most one maximal or minimal element where in (partially) ordered sets there can be multiple maximal or minimal elements.

Example: Let $X := \{a, b, c\}$ and $X_2 := \mathfrak{P}(X) \setminus \{\{\emptyset\}, \{a, b, c\}\}$. Then $\{a\}, \{b\}, \{c\}$ are minimal elements and $\{a, b\}, \{a, c\}, \{b, c\}$ are maximal elements of X_2.

Theorem. *Let $(X, \preceq)$ be a (partial) ordered set.*
*Then there is the **principle of transfinite induction**:*
Let $P(x)$ be a property, s.t.

(1) $P(x_0)$ is true, where x_0 is a minimal element of X,

(2) for arbitrary $x \in X$, $P(x')$ is true $\forall x' \preceq x$ where $x' \neq x$, implies that $P(x)$ is true.

Then $P(x)$ is true $\forall x \in X$.

Proof. "PRINCIPLE OF THE LEAST CRIMINAL"
Suppose $P(x)$ is not true $\forall x \in X$. Then $V := \{x \in X : P(x) \text{ is wrong}\}$, the set of all "criminals", is non-empty. Because of $V \subseteq X$ and the assumption that X is well-ordered by $\preceq$, there is a minimal element v_0 of V. (1) implies that $v_0 \neq x_0$, and (2) implies that $P(x)$ is true $\forall x \in X$ satisfying $x \preceq v_0$, $x \neq v_0$. Moreover it follows that $P(v_0)$ is true. *Contradiction!* $\square$

5 THEOREMS IN CONNECTION WITH THE AXIOM OF CHOICE

Two dual ideas in set theory have to do with finding the "largest" possible objects in some set under a given ordering and with making a simultaneous selection of objects from many sets. These notations are phrased in terms of Zorn's Lemma and the Axiom of Choice.
Although both sound very diffrent, they are equivalent to one another.
The axiom of choice(abbr.AC, named by Zermelo, is an fundamental statement of set theory. It becomes almost indispensable in mathematics, since a large number of important results have been obtained from it across almost all branches of mathematics.
For instance, in linear algebra,the Axiom of Choice guarantees the existence of a basis.
Moreover, it is independent from Zermelo-Fraenkel set theory axioms.
In the following, we are going to focus on the Axiom of Choice and their equivalents. Our purpose in this chapter is to introduce many important propositions,equivalent to the AC, which can be proofed by using the axiom of set theory. First of all, we want to introduce the Axiom of Choice.

5.1 The Axiom of Choice

Axiom of Choice. (Version No. 1)
For every nonempty set $\mathscr{A}$ of nonempty sets, there exists a function

$$f : \mathscr{A} \to \bigcup_{A \in \mathscr{A}} A$$

for which holds for all $A \in \mathscr{A} : f(A) \in A$.

In other words, if we have a bunch of sets, then we can create a new set by selecting one element from each of the given sets.
The function which "chooses" these elements is called *choice function*.
For finite sets we can conclude this without using the Axiom of Choice. Thats why it makes more sense to apply the axiom on infinite sets.

One form of the axiom of choice which was given by Russell in 1906 and Zermelo in 1908 is the following:

Axiom of Choice. (by Zermelo)
Let $\mathscr{A}$ be a disjoint collection of nonempty sets,then there exists a set which consists of one and only one element from each set in $\mathscr{A}$

Example If $\mathscr{A}$ is the collection of all nonempty subsets of $\{1, 2, 3, ...\}$, then we can define f quite easily: just let f(A) be the smallest member of A.

The controversy is how to interpret the words "choose" and "exist" in the axiom:

- If we follow the *constructivists*, and "exist" means "find", then the axiom is false, since we cannot find a choice function for the nonempty subsets of the reals.
- However, most mathematicians give "exists" a much weaker meaning, and they consider the axiom to be true: the defined f(A) "picks any arbitrary member" of A.

The disadvantage is, that the AC only guarantees the existence of such a function, but does not explain how to construct it. That is why many maps, which have to exist due to the AC, cannot be determined. Because of this problem, many mathematicians reject the assertion of the Axiom of Choice.
Now we will present you some important and famous propositions which are connected to

the Axiom of Choice, namely Zorn's Lemma, the Well-ordering Principle and Hausdorff's Maximal Principle.

5.2 Zorn's Lemma

Zorn's Lemma, named after the German mathematician Max August Zorn, who discovered it, is the version often used in applications.
Recall: a totally ordered set is also called a chain.

> **Zorns Lemma.** *If* $(A, <)$ *is a partially ordered set, such that* $A \neq \emptyset$ *and every subset of A simply ordered by* $<$ *has an upper bound,then A has a maximal element under* $<$.

5.3 Well-ordering Principle

Before 1904, when Zermelo published his proof that the Axiom of Choice implies the Well-ordering principle, the Well-ordering Principle was considered self-evident. Cantor and others used it without hesitation. Nevertheless, it is hard to imagine that this principle is true, and moreover, equivalent to the Axiom of Choice.

> **Well-ordering Principle.** Every set can be well-ordered.

5.4 Hausdorff's Maximal Principle

Finally we want to introduce the Hausdorff's Maximal Principle, also called the maximal chain theorem, which is an earlier formulation of Zorn's Lemma, proved by the german mathematician Felix Hausdorff in 1914.
Recall : a subset Q is called maximal totally ordered , if no element of P can be added, without destroying the total order. And a chain is a totally ordered set.

> **Hausdorffs Maximal Principal.** Let (X,<) be a partially ordered nonempty set. Then there is a maximal chain in P.

In the following we want to prove the equivalences of the previous propositions with the Axiom of Choice. Firstly, we show the equivalence of the two versions of the Axiom of Choice.
Next we prove the equivalence relation Zorn's Lemma $\Leftrightarrow$ Hausdorff Maximal Principle and finally the chain of implications
Zorn's Lemma$\Rightarrow$Well-ordering Principle $\Rightarrow$Axiom of Choice$\Rightarrow$ Zorn's Lemma.
Furthermore, the previously represented definitions and axioms are the basis of the following proofs.

Theorem. *The represented versions of the Axiom of Choice are equivalent.*

Proof. AC1 $\Rightarrow$ AC2
Let $\mathscr{A}$ be a nonempty set of pairwise disjoint nonempty subsets. The Axiom of Choice version 1 implies, that there exists a choice function $f : \mathscr{A} \to \bigcup_{A\in\mathscr{A}} A$ with $f(A) \in A \forall \quad A \in \mathscr{A}$. By the axiom of replacement $f(\mathscr{A})$ is a set, which contains $f(A) \in A \forall A \in \mathscr{A}$ because of the property of f, and we are done.

AC2$\Rightarrow$AC1
Let $\mathscr{A}$ be a set of nonempty sets. We want to construct a choice function $f \subseteq \mathscr{A} x \bigcup_{A\in\mathscr{A}} A$. For this we consider the set $M := \{AxA : A \in \mathscr{A}\}$.
It follows that M is a set of nonempty pairwise disjoint sets and the Axiom of Choice version 2 allows us to choose a set f , which consists of one and only one element of each element in M.

We want to verify that f is a choice function by thinking which elements are contained in f. M consists of the cartesian product $\{A\}xA$ of all $A \in \mathscr{A}$, which are also sets. The choice set f has from every $\{A\}xA$ precisely one element, in other words an ordered pair (A, a), where $a \in A$. Since $(A, a) \in \mathscr{A} x \bigcup_{A\in\mathscr{A}} A$, f is as required. ☐

Theorem. *The following principals are equivalent*
(i)Zorn's Lemma
(ii)Hausdorff's Maximal Principal

Proof. (i)$\Rightarrow$(ii)
Let $(X, \leqslant)$ be a nonempty partially ordered set. Since $X \neq \emptyset$, the set of all chains(totally ordered subsets) $\mathscr{K}$ are also not empty. For instance the one element subset of $\mathscr{K}$ is obviously totally ordered. Now we want to apply Zorn's Lemma on $\mathscr{K}$. First of all $\mathscr{K} \subseteq \mathscr{P}(X)$ is, with respect to set inclusion, a partial order. Let K be an arbitrary totally ordered subset of $(\mathscr{K}, \subseteq)$, we consider the set $V := \bigcup_{T\in K} T$. We want to show that V is an upper bound of K in $(X, \leqslant)$. Firstly, it holds $V \in K$ with $x, y \in T$ as $x, y \in V$ are arbitrary choosen. Then there is by construction of V a $T \in K$ with $x, y \in T$. Since K is totally ordered with respect to $\subseteq$ and only has totally ordered subsets with respect to < it follows $x < y$ or $y > x$. So V is totally ordered with respect to <. Next, T is an arbitrary element of K and $t \in K$, it follows $t \in \bigcup_{T\in K} T = V$ so $T \subseteq V$ for all $T \in K$. Hence, V is upper bound of K w.r.t. $\subseteq$. By Zorn's Lemma $(X, \leqslant)$ has a maximal element, so it is a maximally totally ordered subset of X.
(ii)$\Rightarrow$(i)
Let $(X, \leqslant)$ be a partially ordered set, where every totally ordered subset $M \leqslant X$ possesses an upper bound in X. For an arbitrary $x \in X$ we consider the set $Z := y \in X : x \leqslant y$. Since $\leqslant$ is a partially ordering of X, it holds $x \leqslant x$,so $x \in Z$. By Hausdorffs Maximal Principle, there is a maximal chain in Z. By assumption, K posseses an upper bound in X, called m. Then it holds $k \leqslant m$ for all $k \in K \subseteq Z$ that $m \in Z$. We suppose, $m \notin K$ then the set $K' := K \cup \{m\}$ would be totally ordered with $K \subset K'$ in contradiction to the maximality of K, so $m \in K$.

Finally we have to show, that m is **maximally** in X. By contradiction proof we assume that there is an element $a \in X$ with $m \leqslant a$. Consequently, $a \in Z$ and by maximality of K $a \in K$.

Then by the property of an upper bound of m, $a \leqslant m$. Since $\leqslant$ is antisymmetric, it follows $a = m$, so m is maximal in $(X, \leqslant)$.

$\square$

Theorem. *The Axiom of Choice, Zorn's lemma and the Well-ordering Principle are equivalent.*

Proof. Zorn$\Rightarrow$Well-ordering
Let X be a set. We consider the set

$$\mathcal{M} := \{(Y, \leqslant_Y) : Y \subseteq X, \leqslant_Y \text{ a well-ordering}\}$$

Since $(\emptyset, \leqslant_\emptyset)$ (here is $\leqslant_\emptyset$ the empty relation on $\emptyset$) is a well-ordered set, the set $\mathcal{M}$ is not empty.
We define an ordering on M:
Let $(Y, \leqslant_Y) \preceq (Z, \leqslant_Z)$,if Y is an *initial part of* Z ,i.e.

(1) Y$\subseteq$ Z

(2) the restriction of order $\leqslant_Z$ on Y agrees to $\leqslant_Y$ and

(3) is $y \in Y$ and $z \in Z$ with $z \leqslant_Y y$, so $z \in Y$

assumption: The order $\preceq$ on $\mathcal{M}$ is inductive. To that,let $\mathcal{K} \subseteq \mathcal{M}$ be a chain. We consider the subset K:= $\bigcup_{(Y, \leqslant_Y) \in K} Y \subseteq X$ on which we define an order. When k_1,$k_2 \in K$,we can find a $(Y, \leqslant_Y) \in \mathcal{K}$ with $k_1, k_2 \in Y$,since K is a chain. Now we define $k_1 \leqslant_K k_2$, if $k_1 \leqslant_Y k_2$.
If $k_1 \leqslant_Z k_2$ for all $(Z, \leqslant_Z) \in \mathcal{K}$ with $k_1, k_2 \in Z$ hold, can be followed by (2) and the fact that $\mathcal{K}$ is a chain. As every $(Y, \leqslant_Y) \in K$ is a totally ordered set, the order on K is also totally ordered.
It remains to show that $\leqslant_K$ is a well-ordering. For that let $M \subseteq K$ be a nonempty subset. Then there is an element $y \in M$ and we find a $(Y, \leqslant_Y) \in \mathcal{K}$ with $y \in Y$.
Let $m := \min_{\leqslant_Y}(M \bigcap Y)$=$\min_{\leqslant_K}(M \bigcap Y)$(exists because Y is a well-ordering). We assume that $m = \min \leqslant_K(M)$. To that let $z \in M$.Then there is a $(Z, \leqslant_Z) \in \mathcal{K}$ with $z \in Z$. If $Z \subseteq Y$ holfs so $z \in Y \cap M$ and hence $m \leqslant z$. Is that not the case, $Y \subseteq Z$. In particular $m \in Z$. If $m \leqslant z$ doesn't hold,then $z < m$ und by definition of order on $\mathcal{M}$(part(3)) follows $z \in Y$, such that $z \in Y \cap M$ with $z < m$ contradicts the definition of m . So we have shown that $m = \min M$.
Consequently $(K, \leqslant_K)$ is well-ordered und so an upper bound of the chain $\mathcal{K} \in \mathcal{M}$. Thereby it is shown that $(\mathcal{M}, \preceq)$ is inductively ordered. Thus using Zorn's Lemma, we can find a maximal element $(M, \leqslant_M) \in \mathcal{M}$.
It remains to show that $M = X$. If it is not the case, there is a $x \in X \backslash M$. We set $N := M \cup \{x\}$ and define an order on N, which extend the order on M, defining $m \leqslant_N x$ for all $m \in N$.(So x will be added behind the set M). Now we verify easily, that $(N, \leqslant_N) \in M$ and $(M, \leqslant_M) \preceq (N, \leqslant_N) \in M$ holds, a contradiction. Thus $M = X$ and we shown that X can be well ordered.

(ii) Well-ordering$\Rightarrow$ AC
Let $\mathscr{A}$ be a collection of nonempty sets. By the well-ordering theorem, there is a well-ordering on $\bigcup_{A \in \mathscr{A}} A$. We construct a choice function $f : \mathscr{A} \to \bigcup_{A \in \mathscr{A}} A$, which chooses from every $A \in \mathscr{A}$ the mininmal element with respect to the well-ordering and we are done.
(iii)AC$\Rightarrow$ Zorn

Proofsketch: We consider a nonempty inductiv ordered set X with ordered subset M. By a contradiction proof we assume that M has an upper bound and X has no maximal element. The idea is to construct chains with special properties and to use the axiom of choice to get a contradiction. So we can conclude that X must have a maximal element such that Zorn's Lemma holds. For more details see *lecture notes: Höhere Axiome der Mengenlehre by Karl-Hermann Neeb,2009.*

$\square$

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