Categories

Presentation for the proseminar "Category Theory" by Dr. Bartsch

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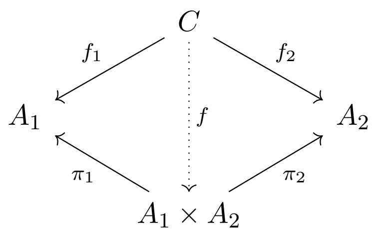
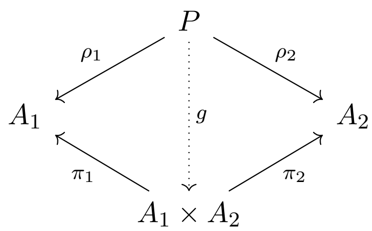
1 Motivation

For the uninformed reader the rather new theory of categories often seems strange. So one may wonder why such a complicated theory is necessary, what we gain by developing it further and how we can apply it to mathematical problems. What becomes clear at first sight is that it is not one of those theories which can be read as an evening read in bed but rather demands concentration on what is known from older subjects like set theory, algebra and topology as well as bringing the notions of these to a more abstract level. The following example should make clear why category theory is a powerful tool.

Consider any two non-empty sets A_1 and A_2 and their cartesian product $A_1 \times A_2$ together with projection functions $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$. The cartesian product has the property that if C is any third set and $f_1 : C \to A_1$ and $f_2 : C \to A_2$ are functions then we can find a unique function $f : C \to A_1 \times A_2$ that satisfies $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$, i.e. the following diagram commutes.



Now the interesting thing is that if there is a set P together with functions $\rho_1 : P \to A_1$ and $\rho_2 : P \to A_2$ having the same property as the cartesian product $A_1 \times A_2$ with π_1 and p_2 then there exists a bijection $g : P \to A_1 \times A_2$ such that $\pi_1 \circ g = \rho_1$ and $\pi_2 \circ g = \rho_2$, i.e. the diagram below commutes.



We can see that P and $A_1 \times A_2$ are identical up to relabeling of elements and so we call the property satisfied by $A_1 \times A_2$ and P universal.

The same can be derived for the direct product of two groups A_1 and A_2 together with projection homomorphism $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$ and also for the topological product $A_1 \times A_2$ of two topological spaces together with continuous projection functions $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$. It follows that if C is any group (topological space, respectively) and $f_1 : C \to A_1$ and $f_2 : C \to A_2$ are homomorphisms (continuous functions) then there is a unique homomorphism (continuous function) $f : C \to A_1 \times A_2$ satisfying $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$. Then again if P is any group (topological space) with projections $\rho_1 : P \to A_1$ and $\rho_2 : P \to A_2$ having the same universal property as the direct product (topological product) then

there exists an isomorphism (homeomorphism) $g : P \to A_1 \times A_2$ such that $\pi_1 \circ g = \rho_1$ and $\pi_2 \circ g = \rho_2$.
We find that the essence of this is the interchangeability of the following terms

$$\begin{array}{ccccc} \text{set} & \longleftrightarrow & \text{group} & \longleftrightarrow & \text{topological space} \\ \text{function} & \longleftrightarrow & \text{homomorphism} & \longleftrightarrow & \text{continuous function} \\ \text{bijection} & \longleftrightarrow & \text{isomorphism} & \longleftrightarrow & \text{homeomorphism} \\ \text{cartesian product} & \longleftrightarrow & \text{direct product} & \longleftrightarrow & \text{topological product} \end{array}$$

by just one column that is more general:

object
morphism
isomorphism
product

2 Categories

2.1 Concrete Categories

We will first give the definition for a concrete category and look at some examples that make it easy to follow the definition. Later on the term abstract category will be defined.

Definition (Concrete Category)**.** A concrete category is a triple $\mathcal{C} = (\mathcal{O}, U, hom)$ where

(i) $\mathcal{O}$ is a class whose members are called $\mathcal{C}$-objects

(ii) $U : \mathcal{O} \to \mathcal{U}$ is a set-valued function, where for each $\mathcal{C}$-object A, $U(A)$ is called the underlying set of A.

(iii) $hom : \mathcal{O} \times \mathcal{O} \to \mathcal{U}$ is a set-valued function, where for each pair (A, B) of $\mathcal{C}$-objects, $hom(A, B)$ is called the set of all $\mathcal{C}$-morphism with domain A and codomain B.

such that the following conditions are satisfied:

(1) For each pair (A, B) of $\mathcal{C}$-objects, $hom(A, B)$ is a subset of the set $U(B)^{U(A)}$ of all functions from $U(A)$ to $U(B)$.

(2) For each $\mathcal{C}$-object A, the identity function $1_{U(A)}$ on the set $U(A)$ is a member of $hom(A, A)$.

(3) For each triple (A, B, C) of $\mathcal{C}$-objects, $f \in hom(A, B)$ and $g \in hom(B, C)$ implies that $g \circ f \in hom(A, C)$ (where '$\circ$' denotes the composition of functions).

To get a more accurate impression of what a concrete category is, we list some important examples:

Examples. 1. **Set**: The class of objects is the class $\mathcal{U}$ (universe) of all sets with $U : \mathcal{U} \to \mathcal{U}$, $U(A) = A$ for all $A \in U$, i.e. U is the identity function and for all $A, B \in \mathcal{U}$, $hom(A, B)$ is the set of all functions from A to B.

2. **Grp**: The class of objects is the class of all groups with $U(A)$ as the underlying set of any group A. All group homomorphism from a group A to a group B form the set $hom(A, B)$.

3. **Top**: The class of objects is the class of all topological spaces with $U(A)$ as the underlying set of any topological space A. All continuous functions from a topological space A to a topological space B form the set $hom(A, B)$.

4. **POS**: The category **POS** has all partially ordered sets as objects together with monotone functions as members of $hom(A, B)$ for any two partially ordered sets A and B. For any $A \in$ **POS**, $U(A)$ is the underlying set.

5. **Ab**: The category **Ab** has all abelian groups as objects together with group homomorpisms as members of $hom(A, B)$ for any two abelian groups A and B. For any $A \in$ **Ab**, $U(A)$ is the underlying set.

6. **SGrp**: This category is built up of all semigroups as the class of objects and has $U(A)$ as the underlying set for any normed semigroup A. The morphisms in $hom(A, B)$ between two semigroups A and B are semigroup homomorphism.

7. **NLinSpace**: This category is built up of all normed spaces as the class of objects and has $U(A)$ as the underlying set for any normed linear space A. The morphisms in $hom(A, B)$ between two normed linear spaces are bounded linear transformations.

2.2 Abstract Categories

Now we come to the notion of abstract categories where we analyze more general structures as for example we neither require morphisms to be functions nor the composition law to be the composition of functions.

Definition (Abstract Category)**.** A category is a quintuple $\mathcal{C} = (\mathcal{O}, \mathcal{M}, dom, cod, \circ)$ where

(i) $\mathcal{O}$ is a class whose members are called $\mathcal{C}$-objects

(ii) $\mathcal{M}$ is a class whose members are called $\mathcal{C}$-morphisms

(iii) dom and cod are functions from $\mathcal{M}$ to $\mathcal{O}$ ($dom(f)$ is called the domain of f and $cod(f)$ is called the codomain of f).

(iv) $\circ$ is a funcion from

$$D = \{(f, g) | f, g \in \mathcal{M} \text{ and } dom(f) = cod(g)\}$$

into $\mathcal{M}$, called the composition law of $\mathcal{C}$ (we say that $f \circ g$ is defined if and only if $(f, g) \in D$).

such that the following conditions are satisfied:

(1) **Matching Condition**: If $f \circ g$ is defined, then $dom(f \circ g) = dom(g)$ and $cod(f \circ g) = cod(f)$;

(2) **Associativity Condition**: If $f \circ g$ and $h \circ f$ are defined, then $h \circ (f \circ g) = (h \circ f) \circ g$

(3) **Identity Existence Condition**: For each $\mathcal{C}$-object A there exists a $\mathcal{C}$-morphism e such that $dom(e) = cod(e)$ and

    (a) $f \circ e = f$ whenever $f \circ e$ is defined

    (b) $e \circ g = g$ whenever $e \circ g$ is defined

(4) **Smallness of Morphism Class Condition**: For any pair (A, B) of $\mathcal{C}$-objects, the class

$$hom_{\mathcal{C}}(A, B) = \{f | f \in \mathcal{M}, dom(f) = A \text{ and } cod(f) = B\}$$

is a set.

Remark. The identity e is not only existent but also unique.

Before giving any examples of abstract categories we will establish three more terms:

Definition. A category $\mathcal{C}$ is said to be:

(1) small provided that $\mathcal{C}$ is a set.

(2) discrete provided that all of its morphisms are identities.

(3) connected provided that for each pair (A, B) of $\mathcal{C}$-objects, $hom_{\mathcal{C}}(A, B) \neq \emptyset$.

Examples. 1. Given a concrete category $\mathcal{C} = (\mathcal{O}, \mathcal{U}, hom)$ there exists a category naturally associated with it, i.e. the class of objects is $\mathcal{O}$, the morphism sets are $hom(A, B)$ and the composition law is the usual composition of functions. It is small/discrete/connected if the concrete category is small/discrete/connected.

2. The category of sets and relations with all sets as objects, all relations from one set A to another set B as members of $hom(A, B)$ and composition law for relations. This category is connected.

3. Given a quasi-ordered class $(\mathcal{C}, \leq)$ ($\mathcal{C}$ is a class with reflexive, transitive relation $\leq$ on $\mathcal{C}$) we gain a category $\tilde{\mathcal{C}}$ by making the objects of $\mathcal{C}$ the objects of $\tilde{\mathcal{C}}$. For the morphisms in $\tilde{\mathcal{C}}$ we demand that $hom(A, B)_{\tilde{\mathcal{C}}}$ contains exactly one element if $A \leq B$ and is empty otherwise. (The same can be done for partially-ordered and totally-ordered classes.) The properties small/discrete/connected depend on the chosen class with its own ordering.

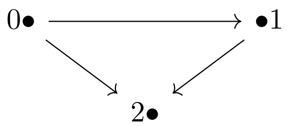
4. Consider the set $\{0, 1, 2..., n-1\}$ for any $n \in \mathbb{N}$ with the usual order of the natural numbers. According to 3. this can be considered a small category and can be visualized for different n as follows:

$0 =$ The empty set

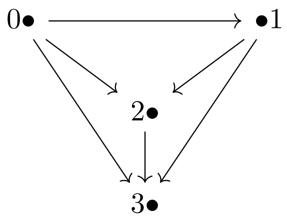
$1 = \bullet$

$2 = 0\bullet \longrightarrow \bullet 1$

$3 =$



$4 =$



These examples give rise to the so called concretizable categories. Each of the four examples is such a concretizable category, namely for each of them there is a concrete category $\tilde{\mathcal{C}}$ such that the category naturally associated with $\tilde{\mathcal{C}}$ is 'isomorphic' with $\mathcal{C}$. Most categories are concretizable, however, we will give an example of a not concretizable category later on when we come to quotient categories.

3 Forming new categories from old ones

We will start this section with a very familiar and intuitive notion. The notion of (full) subcategories.

Definition (subcategory)**.** A category $\mathcal{B}$ is said to be a subcategory of the category $\mathcal{C}$ provided that the following conditions are satisfied:

(1) $\mathcal{O}(\mathcal{B}) \subseteq \mathcal{O}(\mathcal{C})$

(2) $\mathcal{M}(\mathcal{B}) \subseteq \mathcal{M}\mathcal{C})$

(3) The domain, codomain and composition functions of $\mathcal{B}$ are restrictions of the corresponding functions of $\mathcal{C}$.

(4) Every $\mathcal{B}$-identity is a $\mathcal{C}$-identity.

A subcategory $\mathcal{B}$ of $\mathcal{C}$ is said to be a full subcategory provided that for all $A, B \in O(\mathcal{B}), hom_{\mathcal{B}}(A, B) = hom_{\mathcal{C}}(A, B)$.

Examples. (1) Each category is a full subcategory of itself.

(2) The category of finite sets is a full subcategory of **Set**.

(3) The category of sets and injective functions is a subcategory of **Set** but not a full one as surjective and bijective functions are missing (For the category of sets and surjective functions (bijective functions, respectively) the same statement is true).

(4) The category of sets and relations is not a subcategory of **Set** as point (2) of the definition is not satisfied.

3.1 Quotient Categories

Quotient categories are an approach, yet not obvious way, to form new categories from old ones. But in order to give an impression of what a quotient category is it is necessary to know what a congruence on the class of morphisms of a category is.

Definition (congruence)**.** An equivalence relation $\sim$ on the class of morphisms of a category $\mathcal{C}$ is called a congruence on $\mathcal{C}$ provided that:

1. every equivalence class under $\sim$ is contained in $hom(A, B)$ for some $A, B \in \mathcal{O}(\mathcal{C})$, and

2. whenever $f \sim f'$ and $g \sim g'$ it follows that $g \circ f \sim g' \circ f'$ whenever the compositions are meaningful.

Proposition. *If $\sim$ is a congruence on a category $\mathcal{C}$, then the class $\mathcal{D}$ of equivalence classes of morphisms together with the composition law $\tilde{\circ}$ defined by: $\tilde{g}\tilde{\circ}\tilde{f} = \widetilde{g \circ f}$ is a category and we call it the quotient category of $\mathcal{C}$ with respect to $\sim$.*

Remark. In fact a quotient category has essentially the same objects as the category it resulted from.

As mentioned before quotient categories are not always concretizable. This is also true for the next example.

Examples. The category **hTop** is the category **Top** together with an equivalence relation $\sim$ that is defined as follows: For all $A, B \in \mathcal{O}(\mathbf{Top})$ and all $f, g \in hom_{\mathbf{Top}}(A, B)$, $f \sim g$ if and only if f and g are homotopic. Freyd has shown that although **Top** is concretizable **hTop** is not.

3.2 Products and sums of categories

Products as well as sums of categories are more palpable than quotient categories and can be easily formed from other categories.

Definition (product category)**.** If $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ are categories, then the product of the morphism classes

$$\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n$$

together with the composition operation

$$(f_1, f_2, \ldots, f_n) \circ (g_1, g_2, \ldots, g_n) = (f_1 \circ g_1, f_2 \circ g_2, \ldots, f_n \circ g_n)$$

is called the product category of $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$.

Definition (sum category)**.** If $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ are categories, then the disjoint union of the morphism classes

$$\mathcal{M}_1 \amalg \mathcal{M}_2 \amalg \cdots \amalg \mathcal{M}_n$$

together with the composition operation

$$(f, i) \circ (g, j) = (f \circ g, i) \text{ if and only if i=j}$$

is called the sum category of $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$.

3.3 Dual Categories

Another very important way of gaining new categories are dual categories. It is not only the fact that each category has a dual category but it also establishes an important principle: the duality principle.

Definition (dual category)**.** For any category $\mathcal{C} = (\mathcal{O}, \mathcal{M}, dom, cod, \circ)$, the opposite (or dual) category of $\mathcal{C}$ is the category $\mathcal{C}^{op} = (\mathcal{O}, \mathcal{M}, cod, dom, \star)$, where $\star$ is defined by $f \star g = g \circ f$.

Remark. Basically just the domain, codomain functions and composition laws are switched. Also $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

Dual categories make it possible to define a dual concept for every categorical concept, i.e. if P is a property concerning morphisms and objects of a category $\mathcal{C}$, then there is the same property for the opposite category $\mathcal{C}^{op}$, which can be (if it is true for $\mathcal{C}^{op}$) translated into a property P^{op} for the category $\mathcal{C}$; in fact this can be derived for categorical statements:
If S is a categorical statement which holds for all categories, then S^{op} also holds for all categories.

3.4 Comma Categories

At last we obtain new categories by choosing an object of $\mathcal{C}$ and forming the comma category of A over $\mathcal{C}$ (of $\mathcal{C}$ over A, respectively).

Definition (comma category over $\mathcal{C}$)**.** If $\mathcal{C}$ is any category and $A \in \mathcal{O}(\mathcal{C})$, then the comma category of A over $\mathcal{C}$ is the category $(A, \mathcal{C})$ whose objects are those $\mathcal{C}$-morphisms that have domain A and whose morphisms from $A \xrightarrow{f} B$ to $A \xrightarrow{f'} B'$ are those $\mathcal{C}$-morphisms $g : B \to B'$ for which the following triangle commutes

$$\begin{array}{ccc} & A & \\ {}^{f}\swarrow & & \searrow^{f'} \\ B & \xrightarrow{\quad g \quad} & B' \end{array}$$

Composition in $(A, \mathcal{C})$ is defined according to the composition in $\mathcal{C}$.

Definition (comma category over A)**.** If $\mathcal{C}$ is any category and $A \in \mathcal{O}(\mathcal{C})$, then the comma category of $\mathcal{C}$ over A is the category $(\mathcal{C}, A)$ whose objects are those $\mathcal{C}$-morphisms that have codomain A and whose morphisms from $B \xrightarrow{f} A$ to $B' \xrightarrow{f'} A$ are those $\mathcal{C}$-morphisms $g : B \to B'$ for which the following triangle commutes

$$\begin{array}{ccc} B & \xrightarrow{\quad g \quad} & B' \\ {}_{f}\searrow & & \swarrow_{f'} \\ & A & \end{array}$$

Composition in $(\mathcal{C}, A)$ is defined according to the composition in $\mathcal{C}$.

To gain some deeper insight into the theory of categories it is necessary to study special morphisms and objects which will be the content of another presentation.

References

[1] H. Herrlich, G.E. Strecker, *Category Theory*, 3rd Edition, Heldermann Verlag, 2007

[2] Steve Awodey, *Category theory*, 2nd Edition, Oxford University Press, 2010