Special Morphisms and Objects

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February 19, 2015

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1 Preface

In this paper and in our talk, we will introduce the reader to some special kinds of morphisms and their category-theoretical definitions.

For example notions of isomorphisms can be useful in many fields of mathematics, yet there is a simple definition in category-theory, that completely captures the essence of isomorphism. Furthermore there are some generalisations of this concept, that are able to classify morphisms, that are not quite isomorphism but have somehow similar properties.

2 Isomorphisms and Inverses

2.1 Isomorphisms

As stated in the preface, isomorphisms are of special interest in many ways. In category theory however, the definition is different from what we have learned in *Linear Algebra I*, due to the fact that here we are talking about morphisms in a much more general manner.

Definition 1. A morphism $A \xrightarrow{f} B$ is called an *Isomorphism*, if there is a morphism $B \xrightarrow{g} A$, such that $g \circ f = id_A$ and $f \circ g = id_B$.
Then g is called the *inverse* of f.

Example 1. (1) On every object A the identity function id_A is an isomorphism with inverse id_A.

(2) In some categories and especially many constructs (categories of structured sets with structure preserving maps), e.g. **Set** (the category of all sets, that has all functions between them as morphisms) and **Vec** (the category of $\mathbb{R}$-vector spaces with all linear transformations between them), isomorphisms are precisely the bijective morphisms.

(3) However in **Top** (the category of topological spaces with all continuous functions between them as morphisms) there are bijective maps that are not isomorphisms: Let $f : [0, 1) \to \{z \in \mathbb{C} : |z| = 1\}$; $t \mapsto e^{2\pi it}$. Then f is a bijective and continuous map, but it does not have a continuous inverse, which means it is not an isomorphism in **Top**.

2.2 Sections and retractions

Lets look at another example, of a morphism that can be reversed in some sense:

Let $f : \mathbb{Z} \to \mathbb{R}$ with $f(n) = n$, let furthermore $g : \mathbb{R} \to \mathbb{N}$ with $f(x) = \lfloor x \rfloor$ be the floor function.
Then for any $n \in \mathbb{Z}$ we have $g \circ f(n) = g(f(n)) = g(n) = n$, hence $g \circ f = id_{\mathbb{Z}}$, but obviously $f \circ g \neq id_{\mathbb{R}}$, since $Img(f \circ g) = \mathbb{Z} \subsetneq \mathbb{R}$.

To classify these morphisms we need the notion of sections and retractions.

Definition 2. A morphism $A \xrightarrow{f} B$ is called a *section*, if there is a morphism $B \xrightarrow{g} A$, such that $g \circ f = id_A$, then g is called the *left-inverse* of f.
Similarly, f is called a *retraction*, if there is a morphism $B \xrightarrow{h} A$, such that $f \circ h = id_B$, then h is called the *right-inverse* of f.

Example 2. (1) In **Set** the sections are precisely the injective functions and the retractions are the surjective functions.

(2) In **Vec** the sections are precisely the injective homomorphisms and the retractions are precisely the surjective homomorphisms.

Proposition 1. *A morphism is an isomorphism if and only if it is both a section and a retraction.*

Proof. That an isomorphism is a section as well as a retraction follows directly from the definition.
Now let $A \xrightarrow{f} B$ be a section and a retraction, $B \underset{h}{\overset{g}{\rightrightarrows}} A$ such that $g \circ f = id_A$ and $f \circ h = id_B$, then:

$$g = g \circ id_B = g \circ (f \circ h) = (g \circ f) \circ h = id_A \circ h = h$$

and therefore f is an isomorphism. □

Remark: The same proof shows, that the *inverse* of a morphism is unique. Note however, that in general *left* or *right inverses* are not.

3 Mono-, epi- and bimorphisms

Mono-, epi- and bimorphisms give us even weaker assertions, that capture some of the interesting features of "more or less" invertible morphisms, namely left/right cancellable morphisms, that still can be very useful.

Definition 3. A morphism $A \xrightarrow{f} B$ is called a *monomorphism*, if for all pairs of morphisms $C \underset{h}{\overset{g}{\rightrightarrows}} A$ the implication

$$f \circ g = f \circ h \quad \Rightarrow \quad g = h$$

holds true. Then f is said to be *left cancellable*.

A morphism $A \xrightarrow{f} B$ is called an *epimorphism*, if for all pairs of morphisms $B \underset{h}{\overset{g}{\rightrightarrows}} C$ the implication

$$g \circ f = h \circ f \quad \Rightarrow \quad g = h$$

holds true. Then f is said to be *right cancellable*.

Example 3. (1) In **Set** the epimorphisms are exactly the surjective functions, and the monomorphisms are precisely the injective functions.

(2) In **Vec** t.f.a.e.:

a. $f : A \to B$ is a section
b. $f : A \to B$ is an monomorphism
c. $f : A \to B$ is injective

Proof. *a.* $\Rightarrow$ *b.*: We will show this below (see Proposition 2).

b. $\Rightarrow$ *c.*: Let us assume that $f\colon U \to V$ is not injective, i.e. there are vectors $u_1, u_2 \in U$ such that $f(u_1) = f(u_2)$. Let now $h : \mathbb{R}^2 \to U$, with $h(e_1) = u_1$ and $h(e_2) = u_2$ and similarly $k : \mathbb{R}^2 \to U$, with $k(e_1) = u_2$ and $k(e_1) = u_2$. Then we get

$$f(h(e1)) = f(u1) = f(u2) = f(k(e1))$$

and

$$f(h(e2)) = f(u2) = f(u1) = f(k(e2)) \ ,$$

so $f \circ h$ and $f \circ k$ coincide on the standard basis of $\mathbb{R}^2$, which means that these two maps are equal. But of course $k \neq h$, so $f \circ g = f \circ h \not\Rightarrow g = h$, therefore f is not a monomorphism.

c. $\Rightarrow$ *a.*: If $f\colon U \to V$ is injective, then $f\colon U \to f(U)$ is bijective and therefore an isomorphism and has an inverse f^-1. Let now $g\colon B \to A$ with $g(v) = f^-1(v)$ if $v \in image(f)$ and $g \equiv 0$ on some linear complement of $f(A)$. Then for all v in A: $g(f(v)) = v$, i.e. $g \circ f = id_A$. Therefore f is a section. □

Similar we can show that retraction $\Leftrightarrow$ epimorphism $\Leftrightarrow$ surjectivity.

(3) In many other constructs the injective functions are monomorphisms and surjective functions are epimorphisms.

(4) However, there are monomorphisms that are not injective and epimorphisms that are not surjective:

In **Rng** (the category with rings as objects and ring homomorphisms between them as morphisms) the embedding $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism.

Proof. To show this, we look at homomorphisms $h, k : \mathbb{Q} \to \mathbb{Z}$ such that $h \circ f = k \circ f$. Then we have for all $\frac{n}{m} \in \mathbb{Q}$:

$$\begin{aligned} h\left(\frac{n}{m}\right) &= h(n) \cdot h\left(\frac{1}{m}\right) \cdot h(1) \\ &= k(n) \cdot h\left(\frac{1}{m}\right) \cdot k(1) \\ &= k(n) \cdot h\left(\frac{1}{m}\right) \cdot k(m) \cdot k\left(\frac{1}{m}\right) \\ &= k(n) \cdot h\left(\frac{1}{m}\right) \cdot h(m) \cdot k\left(\frac{1}{m}\right) \\ &= k(n) \cdot h(1) \cdot k\left(\frac{1}{m}\right) \\ &= k\left(\frac{n}{m}\right) \end{aligned}$$

□

Definition 4. A morphism, that is both a monomorphism and an epimorphism is called a *bimorphism*

Example 4. (1) Following the examples from above, the bimorphisms in **Set** and **Vec** are exactly the isomorphisms.

(2) However, there are bimorphisms that are not isomorphisms, e.g. the embedding of the integers into the rational numbers $f : \mathbb{Z} \to \mathbb{Q}$, $z \mapsto z$ is a non-isomorphic bimorphism in **Rng**: In the previous example we saw that f is an epimorphism. As f is injective, f is also a monomorphism, hence a bimohrphism. But f is not bijective, therefore f cannot be an isomorphism.

Definition 5. A category is called *balanced* provided that any bimorphism is an isomorphism

Remark: Following the last example the categories **Set** and **Vec** are balanced, while **Rng** is not.

3.1 Correlations

In the examples from above we have already seen that in some cases (e.g. **Vec** and **Set**) sections and monomorphisms coincide, while in other cases (e.g. the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in **Rng**) this is not true. Therefore we are now looking at general relations between the different kinds of morphisms we looked at above.

Proposition 2. *Any section is a monomorphism.*

Proof. Let $A \overset{f}{\to} B$ be a section, and $B \overset{g}{\to} A$ its left inverse. Then, for any pair of morphisms $C \underset{h_2}{\overset{h_1}{\rightrightarrows}} A$ we obtain:

$$f \circ h_1 = f \circ h_2 \Rightarrow g \circ f \circ h_1 = g \circ f \circ h_2 \Rightarrow h_1 = h_2$$

□

Remark: Similarly any retraction is an epimorphism, and putting these two results together, we obtain that any isomorphism is a bimorphism.

In the second section we saw that a morphism, which is both a section and a retraction, is an isomorphism. But in fact we can weaken the assumptions for this proposition to hold:

Proposition 3. *A morphism is an isomorphism if and only if it is both a retraction and a monomorphism.*

Proof. The forward direction is again clear. Let $A \xrightarrow{f} B$ be a monomorphism and $B \xrightarrow{g} A$ its left inverse, i.e. $f \circ g = id_B$.
Then we have

$$f \circ (g \circ f) = (f \circ g) \circ f = id_B \circ f = f = f \circ id_A$$

and therefore, by left cancellation, $(g \circ f) = id_A$ □

Remark: Similarly a morphism is an isomorphism if and only if it is both a section and an epimorphism.

4 References

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