Limits and Colimits

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1 Limits and Colimits

In category theory, limits are important notions to generalize various concepts such as equalizers, products and pullbacks. A limit combines elements of a category that are somehow related.

For example, we will see that a collection of sets can be combined to their Cartesian product. Another example is the intersection of the kernel of two group homomorphisms between the same groups.

For colimits, the same holds as well. As those two notions are very similar, we will not expand on colimits as much as on limits. But as colimits are not less important, it is also worthy mentioning them.

1.1 Diagrams

Category theory often makes use of diagrams. For limits and colimits we use a formalized definition of them.

Definition 1.1. *Let $\boldsymbol{E}$ be a small category and C a category. A* ***diagram*** *of type $\boldsymbol{E}$ is a functor $F : \boldsymbol{E} \to \boldsymbol{C}$.*

Intuitively, a diagram selects some objects and morphisms in a category $\mathbf{C}$. That is the same as diagrams in a graphical sence do. They choose some objects and morphisms of a category.

We can think of $\mathbf{E}$ as being an index category. In our use, the actual objects and morphisms in $\mathbf{E}$ are not relevant, only the form of the image of the functor matters.

1.2 Limits

To form the notion of limits, we initially create the concept of cones.

Definition 1.2. *Let $F : \boldsymbol{E} \to \boldsymbol{C}$ be a diagram. A* ***cone with vertex L*** *is a pair $(L, \lambda_i)_{i \in Ob(\boldsymbol{E})}$ for an object L of C and a family of morphisms $\lambda_i : L \to F(i)$, such that for every morphism $f : i \to j$ in $\boldsymbol{E}$ we have $F(f)\lambda_i = \lambda_j$.*

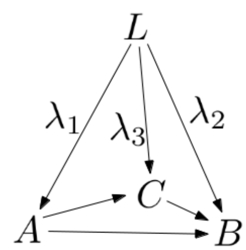

Figure 1: Example for a cone

As you can see in Figure **??**, we call such a pair a cone because the vertex L together with the morphisms λ_i and $F(i)$ can be geometrically illustrated as a cone.

Now we can define a limit as a special type of cone.

Definition 1.3. *A cone* $(L, \lambda_i)_{i \in Ob(\boldsymbol{E})}$ *is called a* ***limit*** *if for every cone* $(L', \lambda'_i)_{i \in Ob(\boldsymbol{E})}$ *there exists a unique morphism* $l : L' \to L$ *such that* $\lambda'_i = \lambda_i l$.

Figure **??** shows a limit. Using that diagram, we can define a limit just as a cone such that the diagram commutes for every $(L', \lambda'_i)_{i \in Ob(\mathbf{E})}$ and l is unique.

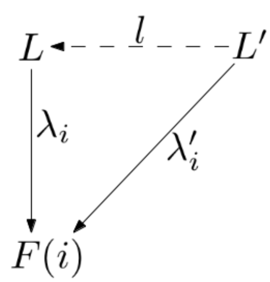

Figure 2: Limit

That is, a limit of a diagram is a cone such that every other cone of the same diagram can be uniquely factorized into a morphism and the limit.

We already know limits for metric spaces from analysis. It is possible to transform those limits into limits in the sense of category theory, so that both concepts are equivalent.

But we now look at another rather simple example.

Example 1.1. *Let* $\boldsymbol{C}$ *be the partial order category of sets. That means, we define* $Ob(\boldsymbol{C})$ *to be the class of all sets. The morphisms are defined such that for all* $a, b \in Ob(C) |hom(a, b)| = 1$ *if* $a \subset b$ *and* $|hom(a, b)| = 0$ *otherwise.*

Then let $F : \boldsymbol{E} \to \boldsymbol{C}$ *be a diagram. Now we can obtain a cone* $(L, \lambda_i)_{i \in Ob(\boldsymbol{E})}$*, with the intersection* $L := \cap_{A \in \boldsymbol{D}} F(A)$ *and* $\lambda_i : L \to F(i)$. λ_i *are the unique morphisms that indicate the subset relation.*

Now we want to proof that this cone is a limit. Let $(L', \lambda'_i)_{i \in Ob(\boldsymbol{E})}$ be another cone. We need to show that there exists a unique morphism $l : L' \to L$ such that $\lambda'_i = \lambda_i l$.

- *Existence: Every set that is a subset of all sets in $F(i)$ is a subset of the intersection L. So there is a morphism from L' to L.*
- *Uniqueness: By our definition of $\boldsymbol{C}$, there is no more than one morphism between two objects. So that morphism is unique.*

In that example, we can nicely see that sometimes of a limit can be thought of like an infimum. A set that is subset of all our sets in the diagram F is a lower bound in terms of the subset order. This lower bound forms a cone. Then, the intersection of the sets is the greatest lower bound or infimum.

1.3 Colimits

Looking at diagrams of Cones and Limits, we see that the morphisms from the vertex of our cone are directed in a fixed direction. That leads to the question, if we can reverse the directions. Indeed we can.

Reversing the direction of the morphisms in our definition of cones and limits will give us the dual notion of them. Those notions are called *co-cones* and *colimits*.

Definition 1.4. *Let $F : \boldsymbol{E} \to \boldsymbol{C}$ be a diagram. A* ***co-cone with vertex L*** *is a pair $(L, \lambda_i)_{i \in Ob(\boldsymbol{E})}$ for an object $L \in Ob(\boldsymbol{C})$ and a family of morphisms $\lambda_i : F(i) \to L$, such that for every morphism $f : j \to i$ in E we have $\lambda_i F(f) = \lambda_j$.*

Definition 1.5. *A co-cone $(L, \lambda_i)_{i \in Ob(\boldsymbol{E})}$ is called a* ***colimit*** *if for every cone $(L', \lambda'_i)_{i \in Ob(\boldsymbol{E})}$ there exists a unique morphism $l : L' \to L$ such that $\lambda'_i = \lambda_i l$.*

We do not extend further on this topic but rather look at some special types of limits.

2 Products and Equalizers / Coproducts and Coequalizers

2.1 Products and Coproducts

Firstly, we enlarge upon products. We obtain a product by creating a diagram $F : \mathbf{D} \to \mathbf{C}$ over a discrete small category $\mathbf{D}$, i. e. over a category containing just a set of objects and their identities. As such a diagram has no morphisms except for identities, we can assume it equivalently just to be a familiy of objects in $\mathbf{C}$. That gives us a simpler kind of a limit:

Definition 2.1. *Let $(A_i)_{i \in I} \subset \boldsymbol{C}$ be a family of objects in a category $\boldsymbol{C}$ and I an index set. A* ***product*** *is a pair $(L, \lambda_i)_{i \in I}$ for an object L of $\boldsymbol{C}$ and a family of morphisms $\lambda_i : L \to A_i$, such that for every other pair $(L', \lambda'_i)_{i \in I}$ there exists a unique morphism $l : L' \to L$ such that $\lambda'_i = \lambda_i l$.*

An example of a product is the Cartesian product for sets:

Example 2.1. *Let* $\boldsymbol{C} := \boldsymbol{Set}$ *be the category of all sets with morphisms representing the maps between those sets. Furthermore, let* $A_i, i \in M$ *for some finite* $M := \{1, 2, ..., n\}$ *be a family of sets.*

Then the Cartesian product $L := \prod_{i \in M} A_i = A_1 \times A_2 \times ... \times A_n$ *of the sets is also in* $\boldsymbol{C}$*. For every* $i \in M$*, we associate the map* λ_i*, that maps the i-th component of* L *identically to* A_i*. By calling that component* L_i*, we can write it as* $\lambda_i(x) = x_i$ *for* $x \in L$*.*

We want to show that $(L, \lambda_i)_{i \in M}$ *is a product. So, let* $(L', \lambda'_i)_{i \in M}$ *be another family. We need to show that there exists a unique morphism* $l : L' \to L$ *such that* $\lambda'_i = \lambda_i l$*:*

- *Existence: For some* $x \in L'$ *we define* $l(x) = \prod_{i \in M} \lambda'_i(x)$*. Then we have* $\lambda_i(l(x)) = \lambda_i(\prod_{j \in M} \lambda'_j(x)) = (\prod_{j \in M} \lambda'_j(x))_i = \lambda'_i(x)$ *for all* $i \in M$*. Hence our composition condition is satisfied.*

- *Uniqueness: Assume there exists another morphism* $l' : L' \to L$ *with* $l' \neq l$*. Then there is some* $x \in L'$ *such that there is at least one component* $i \in M$ *with* $(l(x))_i \neq (l'(x))_i$*.*

  Then we deduce for that component $\lambda_i(l'(x)) = (l'(x))_i \neq (l(x))_i = (\prod_{j \in M} \lambda'_j(x))_i = \lambda'_i(x)$*. Hence we have* $\lambda'_i \neq \lambda_i \circ l'$*. So the composition condition is not satisfied. Therefore,* l *is unique.*

As we have already seen for colimits, we can also swap the direction of the morphisms to obtain the notion of coproducts:

Definition 2.2. *Let* $(A_i)_{i \in I} \subset \boldsymbol{C}$ *be a family of objects in a category* $\boldsymbol{C}$ *and* I *an index set. A* ***coproduct*** *is a pair* $(L, \lambda_i)_{i \in I}$ *for an object* L *of* $\boldsymbol{C}$ *and a family of morphisms* $\lambda_i : A_i \to L$*, such that for every other such pair* $(L', \lambda'_i)_{i \in I}$ *there exists a unique morphism* $l : L \to L'$ *such that* $\lambda'_i = l\lambda_i$*.*

2.2 Equalizers and Coequalizers

Another special type of limits are equalizers. We obtain a equalizer by creating a limit of a diagram like $\cdot \rightrightarrows \cdot$, that is, a diagram containing just two objects with their identities and two parallel morphisms f and g between them.

Definition 2.3. *Let* $\boldsymbol{C}$ *be a category and* $f, g : X \to Y$ *be morphisms. A pair* (E, e) *with* $E \in Ob(\boldsymbol{C})$ *and* $e \in Hom_{\boldsymbol{C}}(E, X)$ *is called an* ***equalizer*** *of* f *and* g *if* $fe = ge$ *and for each other such pair* (Z, h) *there exists a unique morphism* $h' : Z \to E$*.*

As E *is already determined by the morphism* e*, it is common to call* e *itself an equalizer.*

In other words, an equalizer e is a morphism that makes two morphisms equal by composing them with e. Figure **??** shows an equalizer. A morphism e is an equalizer if that diagram commutes with an unique h' for every h.

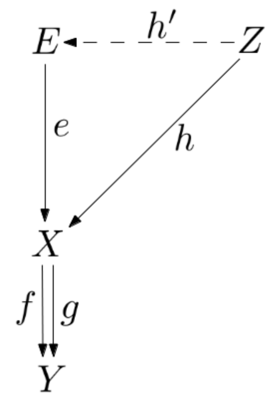

Figure 3: Equalizer

Example 2.2. *As an example, we can look at the category **Grp** of all groups with their homomorphisms. Assume f and g are two homomorphisms with the same domain and codomain. The intersection of their kernels again forms a group (and would in many cases just contain one element). Then the inclusion from the kernel into the domain forms an equalizer.*

Again, we obtain the dual notion by reversing the direction of our morphisms.

Definition 2.4. *Let $\boldsymbol{C}$ be a category and $f, g : X \to Y$ be morphisms. A pair (E, e) with $E \in Ob(\boldsymbol{C})$ and $e \in Hom_{\boldsymbol{C}}(X, E)$ is called an **coequalizer** of f and g if $ef = eg$ and for each other such pair (Z, h) there exists a unique morphism $h' : E \to Z$.*

3 Completeness and Cocompleteness

Now we have seen some examples of limits. But we have not treated the question, when limits exist generally. We will not search for answers for that question, but introduce the notion of completeness. That notion states if limits for a category exist for any diagram.

Definition 3.1. *A Category is **D-(co)complete** if every diagram of type **D** has a limit.*

Definition 3.2. *A Category is **(co)complete** if it is **D**-(co)complete for every small category **D**.*

Those two definitions form the basis for further analysis of the existence of limits.

References

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[2] Preuß, G.: *Grundbegriffe der Kategorientheorie*, Mannheim [u.a.] : Bibliogr. Inst., 1975