Representable Functors

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1 Introduction

This write-up will deal with functors into the category **Set** of all sets, as they have some special properties. They are useful since we can extract knowledge about the structure of **Set** and apply it to many other categories by usage of e.g. preservance of limits, as discussed later. This becomes of special importance for non-concretisable categories.

2 Hom Functors

The so-called hom-functors provide essential functors for any category **A** into **Set** by mapping objects of the category to sets of morphisms. In this write-up we will only deal with covariant hom-functors. A construction of contravariant hom-functors can be done as well, c.f. [1]. Formally, covariant hom-functors are defined as following:

Definition 1. *Let* $\boldsymbol{A}$ *be a category and* A *be an* $\boldsymbol{A}$*-object. The* ***(covariant) hom-functor*** $\hom(A, -) : \mathbf{A} \to \mathbf{Set}$ *is defined by*

$$\hom(A, -)(C \xrightarrow{f} B) := \hom(A, C) \xrightarrow{\hom(A,f)} \hom(A, B)$$

where $\hom(A, f)(g) := f \circ g$ *for* $A \xrightarrow{g} C$.

Examples

- Consider the category **Mat** with objects all natural numbers and morphisms $\hom(m, n) := \mathbb{R}^{n\times m}$ all real-valued matrices with the proper dimensions between them and the identities $id_n := E_n$, where E_n is the $n \times n$ unit matrix. Then the hom-functor $\hom(a, -)$ for any given number $a \in \mathbb{N}$ maps each natural number n to the set $\hom(a, n) = \mathbb{R}^{n\times a}$ of all real-valued $n \times a$ matrices. For any morphism $f : m \to n$, i.e. a matrix $f \in \mathbb{R}^{n\times m}$ the image $\hom(a, f)$ is the linear map from $\hom(a, m) = \mathbb{R}^{m\times a}$ to $\hom(a, n) = \mathbb{R}^{n\times a}$ given by $(\hom(a, f))(A) = f \cdot A$ for $A \in \mathbb{R}^{m\times a}$.

- Consider the following small category $\mathbf{C}$, whose universe consists of the two sets $A := \{0\}$ and $B := \{0, 1\}$. It has (taking away the identity functions) three morphisms f, g, h, which are all constant zero functions.

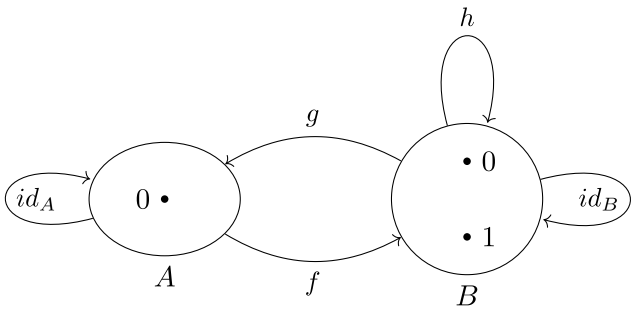

Figure 1: the category $\mathbf{C}$

We obtain the following image under application of $F := \hom(A, -)$ or $G := \hom(B-)$, respectively.

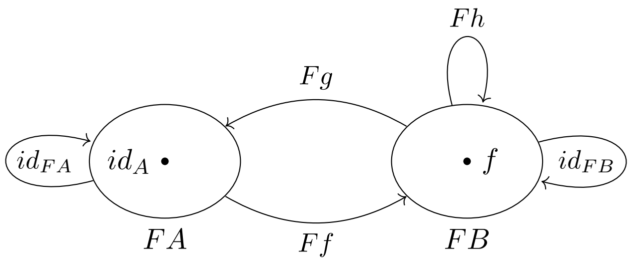

Figure 2: the image $F(\mathbf{C})$

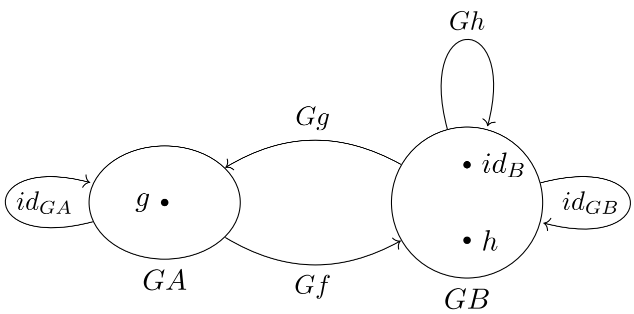

Figure 3: the image $G(\mathbf{C})$

Theorem 1. *Let* $F : \mathbf{A} \to \mathbf{B}$ *be a diagram,* $L \in Obj(\mathbf{B})$ *and* $(l_A : L \to F(A))_{A \in Obj(\mathbf{A})}$ *be a family of* $\mathbf{B}$*-morphisms. Then the following are equivalent:*

(1) $(L, (l_A)_{A \in Obj(\mathbf{A})})$ *is a limit of* F*,*

(2) For each $X \in Obj(\mathbf{B})$ *the pair* $(\hom(X, -)(L), (\hom(X, -)(l_A)))$ *is limit of* $\hom(X, -) \circ F$*.*

For a proof, see [1]. This immediately establishes the following

Corollary 2. *The covariant hom-functors* $\hom(X, -)$ *preserve limits.*

Preservance of limits proves to be useful in cases where we know limits of the domain of a functor as we can then construct limits of the codomain as well.

3 Representable Functors

Since hom-functors are the most natural functors into **Set**, we will now investigate functors which are naturally isomorphic to one of them. We call those functors representable.

Definition 2. *A functor* $F : \mathbf{B} \to \mathbf{Set}$ *is called* ***representable*** *given that it is naturally isomorphic to a covariant hom-functor* $\hom(X, -)$ *for a suitable* $X \in Obj(\mathbf{B})$*. A* ***representation*** *of* F *is a pair* (X, τ)*, where* X *is a* $\mathbf{B}$*-object and* $\tau = (\tau_B) : \hom(X, -) \to F$ *is a natural isomorphism.*

Examples

- The forgetful functor of the category **Vec** of real vector spaces is represented by any one dimensional vector space, e.g. $\mathbb{R}$ itself.
- Similarly, the functor $F : \mathbf{Mat} \to \mathbf{Set}$, mapping n to $\mathbb{R}^n$ is represented by 1.
- The identity functor of the category with objects $\{0\}$, $\{0, 1\}$ and
  - morphisms all constant-zero-mappings[1] between them is represented by $\{0, 1\}$.
  - morphisms all constant-one-mappings[1] is not representable.
  - morphisms all constant mappings[1] is represented by $\{0\}$.
- More generally, the identity functor of any category **C** with objects any class of non-empty sets and functions all constant mappings[1], is represented by any singleton[2].
- The power set functor is not representable.

[1] And the identities.
[2] A set with one element

Proof. Let $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ be the power set functor. Assume $\mathcal{P}$ is representable by some pair (A, τ). Let S be an arbitrary fixed singleton set with $S \neq A$. Then $\tau_S : \hom(A, S) \to \mathcal{P}(S)$ is a bijective map, since τ is a natural isomorphism. But $|\hom(A, S)| = 1$ and $|\mathcal{P}(S)| = 2$ since S is a singleton. Hence by cardinality τ_S cannot be bijective which is $\mathcal{P}$ is not representable. □

One can prove that for two naturally isomorphic functors $F, G : \mathbf{A} \to \mathbf{B}$ the first functor F preserves limits if, and only if, the functor G does so (c.f. [1], p.225). With this knowledge we immediately obtain the following

Corollary 3. *Representable functors preserve limits.*

4 Yoneda Lemma

The Yoneda lemma is useful when dealing with representable functors. First we will present this theorem in a slightly different manner.

Theorem 4. *For any functor $F : \mathbf{A} \to \mathbf{Set}$, any $\mathbf{A}$-object A and any element $a \in F(A)$, there exists a unique natural transformation $\tau : \hom(A, -) \to F$ with $\tau_A(id_A) = a$.*

Proof. To show existence, we define τ by $\tau_C : f \mapsto (Ff)(a)$ for $f \in \hom(A, C)$. It is easy to see that then $\tau_A(id_A) = (F(id_A))(a) = id_{F(A)}(a) = a$. For naturality, we want the diagram

$$\begin{array}{ccc} \hom(A,B) & \xrightarrow{\tau_B} & F(B) \\ \downarrow{\scriptstyle hom(A,f)} & & \downarrow{\scriptstyle F(f)} \\ \hom(A,C) & \xrightarrow{\tau_C} & F(C) \end{array}$$

to commute for all $B, C \in Obj(\mathbf{A})$ and all $f : B \to C$. This is the case since

$$\begin{aligned} (Ff \circ \tau_B)(g) &= Ff(\tau_B(g)) = Ff(Fg(a)) = (Ff \circ Fg)(a) \\ &= (F(f \circ g))(a) = \tau_C(f \circ g) = \tau_C(\hom(A, f)(g)) \\ &= (\tau_C \circ (\hom(A, -)f))(g) \end{aligned}$$

holds for all $g : A \to B$.

For uniqueness, consider any natural transformation $\delta : \hom(A, -) \to F$ with $\delta_A(id_A) = a$. For any $C \in Obj(\mathbf{A})$ and $f : A \to C$ naturality of δ implies

$$\begin{aligned} \delta_C(f) &= \delta_C(f \circ id_A) = (\delta_C \circ \hom(A, f))(id_A) = (Ff \circ \delta_A)(id_A) \\ &= Ff(a) = \tau_C(f). \end{aligned}$$

Thus $\tau = \delta$. □

Remembering $[\mathbf{A}, \mathbf{B}]$ is the functor quasi-category allows us to rephrase the above theorem to:

Corollary 5 (Yoneda Lemma)**.** *Let* $F : \mathbf{A} \to \mathbf{Set}$ *be a functor,* $B \in Obj(\mathbf{A})$ *and* $\mathbf{Q} = [\mathbf{A}, \mathbf{Set}]$*. Then the Yoneda mapping*

$$Y : \hom_{\mathbf{Q}}(\hom_{\mathbf{A}}(B, -), F) \to F(B)$$

defined by $Y(\eta) := \eta_B(id_B)$ *for all* $\eta = (\eta_A)_{A \in \mathbf{A}} \in \hom_{\mathbf{Q}}(\hom_{\mathbf{A}}(B, -), F)$ *is a bijection.*

This lemma has its uses, among many, in showing that we have an embedding of any $\mathbf{A}$ into $[\mathbf{A}^{\text{op}}, \mathbf{Set}]$.

5 Universal Points

Definition 3. *Let* $F : \mathbf{A} \to \mathbf{Set}$ *be a functor. A pair* (U, u) *consisting of an* $\mathbf{A}$*-object* U *and a point* $u \in F(A)$ *is called* ***universal point*** *if for each pair* (A, a) *with* $A \in Ob(\mathbf{A})$ *and* $a \in F(A)$ *there exists exactly one* $f : U \to A$ *with* $(Ff)(u) = a$*.*

Examples

- The pair $(\mathbb{R}, 1)$ is a universal point of the forgetful functor of **Vec**.
- The identity functor on $\mathbf{P}$ with $Obj(\mathbf{P}) = \{\{0\}, \{0, 1\}\}$ and morphisms all constant-zero-maps has the universal point $(\{0, 1\}, 1)$.
- For the identity functor of $\mathbf{K}$ as in the next theorem, $(\{s\}, s)$ is a universal point, if $\{s\} \in Obj(\mathbf{K})$.

Proposition 6. *Let* $\mathbf{K}$ *be a category with objects a class of non-empty sets, morphisms all constant functions between them and the identities on each set. Then for any* $S := \{s\} \in Obj(\mathbf{K})$ *the pair* (S, s) *is an universal point of the identity functor* F*.*

Proof. For any $A \in Obj(\mathbf{K})$ and $a \in FA$ there are two possibilities:

- If $A = S$, we have $a = s$ and there is indeed exactly one morphism fulfilling $a = f(s) = Ff(s)$, namely $id_A = id_S$, which is the only constant function from S into itself, thus satisfying the requirement.
- If $A \neq S$, there is by definition of $\mathbf{K}$ the constant-a-morphism $f : S \to A$, fulfilling $a = f(s) = Ff(s)$. Any other morphism $f \neq g : S \to A$ would need to differ on the only element in the domain[3], thus yielding $a \neq g(s) = Fg(s)$, hence f is unique.

Thus such a unique morphism exists for all (A, a) and (S, s) is a universal point. □

Theorem 7. *Let* $F : \mathbf{B} \to \mathbf{Set}$ *be a functor,* $U \in Obj(\mathbf{B})$ *and* $\eta : \hom(U, -) \to F$ *be a natural transformation. Then the following are equivalent:*

- (U, η) *is a representation of* F*,*

[3]Due to the constantness, this holds for non-singleton S as well. For non singleton sets, this proof would only fail in the first case.

- $(U, Y(\eta))$ *is a universal point of* F.

Proof. We first establish for $A \in Obj(\mathbf{B})$ and $f : U \to A$, that the diagram

$$\begin{array}{ccc} \hom(U,U) & \xrightarrow{\eta_U} & F(U) \\ \downarrow{\scriptstyle \hom(U,f)} & & \downarrow{\scriptstyle Ff} \\ \hom(U,A) & \xrightarrow{\eta_A} & F(A) \end{array}$$

commutes by naturality of η and thus

$$\begin{aligned} \eta_A(f) &= \eta_A(f \circ id_U) = \eta_A(\hom(U,f)(id_U)) = Ff(\eta_U(id_U)) \\ &= Ff(Y(\eta)) \end{aligned}$$

holds. Now, that (U, η) is a representation of F is equivalent to the claim that for all $A \in Obj(\mathbf{B})$ the mapping $\eta_A : \hom(U, A) \to FA$ is a bijection, i.e. for all such A and all $a \in FA$ there is exactly one $f : U \to A$ such that $a = \eta_A(f) = Ff(Y(\eta))$. The last statement however is by definition equivalent to the statement that $(U, Y(\eta))$ is a universal point of F. $\square$

This theorem establishes the following useful corrollary:

Corollary 8. *For each functor* $F : \mathbf{B} \to \mathbf{Set}$ *the following are equivalent:*

- *The functor* F *is representable,*
- *There are universal points of* F.

This corollary is useful when proving (un-)representability of functors.

Corollary 9. *The identity functor of* $\mathbf{K}$ *as in proposition 6 is representable.*

References

[1] Jiří Adámek, Horst Herrlich, and George E. Strecker. *Abstracts and concrete categories : the joy of cats.* Pure and applied mathematics. Wiley, New York [u.a.], 1990.

[2] Gerhard Preuß. *Grundbegriffe der Kategorientheorie*, volume 739 of *B.I-Hochschultaschenbücher.* Bibliogr. Inst., Mannheim [u.a.], 1975.